is as defined in (2.1.11) and

$$\mathfrak{N}_N(\varphi) = \int_{\mathbf{R}^n} (1+|x|)^N \sum_{|\alpha|\le N+1} |\partial^\alpha \varphi(x)|\, dx .$$

We note that as in the scalar case, we have the sequence of simple inequalities

$$M(\vec{f};\Phi) \le M_a^*(\vec{f};\Phi) \le (1+a)^b M_b^{**}(\vec{f};\Phi). \tag{2.1.55}$$

We now define the vector-valued Hardy space $H^p(\mathbf{R}^n,\ell^2_L)$.

Definition 2.1.12. Let $\vec{f}=\{f_j\}_{j=1}^L$ be a finite sequence of bounded tempered distributions on $\mathbf{R}^n$ and let $0<p<\infty$. We say that $\vec{f}$ lies in the vector-valued Hardy space $H^p(\mathbf{R}^n,\ell^2_L)$ if the *Poisson maximal function*

$$M(\vec{f};P)(x) = \sup_{t>0} \big\| \{(P_t * f_j)(x)\}_j \big\|_{\ell^2_L}$$

lies in $L^p(\mathbf{R}^n)$. If this is the case, we set

$$\big\|\vec{f}\big\|_{H^p(\mathbf{R}^n,\ell^2_L)} = \big\|M(\vec{f};P)\big\|_{L^p(\mathbf{R}^n)} = \Big\| \sup_{\varepsilon>0} \Big(\sum_{j=1}^L |f_j * P_\varepsilon|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbf{R}^n)}.$$

The next theorem provides a vector-valued analogue of Theorem 2.1.4.

Theorem 2.1.13. *Let $0<p<\infty$, $L\in\mathbf{Z}^+$. Then the following statements are valid:*
(a) There exists a Schwartz function Φ^o with $\int_{\mathbf{R}^n}\Phi^o(x)\,dx=1$ and a constant C_1 ($C_1=500$ works) such that

$$\big\|M(\vec{f};\Phi^o)\big\|_{L^p(\mathbf{R}^n)} \le C_1 \big\|\vec{f}\big\|_{H^p(\mathbf{R}^n,\ell^2_L)} \tag{2.1.56}$$

for every sequence $\vec{f}=\{f_j\}_{j=1}^L$ of bounded tempered distributions.
(b) For every $a>0$ and Φ in $\mathscr{S}(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n}\Phi(y)dy\neq 0$ there exists a constant $C_2(n,p,a,\Phi)$ such that

$$\big\|M_a^*(\vec{f};\Phi)\big\|_{L^p(\mathbf{R}^n)} \le C_2(n,p,a,\Phi)\big\|M(\vec{f};\Phi)\big\|_{L^p(\mathbf{R}^n,\ell^2_L)} \tag{2.1.57}$$

for every sequence $\vec{f}=\{f_j\}_{j=1}^L$ of tempered distributions.
(c) For every $a>0$, $b>n/p$, and Φ in $\mathscr{S}(\mathbf{R}^n)$ there exists a constant $C_3(n,p,a,b)$ such that

$$\big\|M_b^{**}(\vec{f};\Phi)\big\|_{L^p(\mathbf{R}^n)} \le C_3(n,p,a,b)\big\|M_a^*(\vec{f};\Phi)\big\|_{L^p(\mathbf{R}^n,\ell^2_L)} \tag{2.1.58}$$

for every sequence $\vec{f}=\{f_j\}_{j=1}^L$ of tempered distributions.
(d) For every $b>0$ and Φ in $\mathscr{S}(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n}\Phi(x)\,dx\neq 0$ there exists a constant $C_4(b,\Phi)$ such that if $N=[\frac{n}{p}]+1$ we have

$$\big\|\mathscr{M}_N(\vec{f})\big\|_{L^p(\mathbf{R}^n)} \le C_4(b,\Phi)\big\|M_b^{**}(\vec{f};\Phi)\big\|_{L^p(\mathbf{R}^n,\ell^2_L)} \tag{2.1.59}$$

for every sequence $\vec{f}=\{f_j\}_{j=1}^L$ of tempered distributions.