[*Hint:* The first inequality follows by Leibniz's rule. Conversely, to express $\xi^\alpha \partial^\beta \varphi$ in terms of linear combinations of $\partial^\beta(\xi^\gamma \varphi(\xi))$, proceed by induction on $|\alpha|$, using that $\xi_j \partial^\beta \varphi = \partial^\beta(\xi_j \varphi) - \beta_j \partial^{\beta - e_j} \varphi$ if $\beta_j \geq 1$ and $\xi_j \partial^\beta \varphi = \partial^\beta(\xi_j \varphi)$ if $\beta_j = 0$. Here $\beta = (\beta_1, \dots, \beta_n)$ and $e_j = (0, \dots, 1, \dots, 0)$ with 1 in the jth entry.]

1.1.2. Suppose that a function φ lies in $\mathscr{C}^\infty(\mathbf{R}^n \setminus \{0\})$ and that for all multi-indices α there exist constants L_α such that φ satisfies

$$\lim_{t \to 0} \partial^\alpha \varphi(t) = L_\alpha .$$

Then φ lies in $\mathscr{C}^\infty(\mathbf{R}^n)$ and $\partial^\alpha \varphi(0) = L_\alpha$ for all multi-indices α.

1.1.3. Let $u_N \in \mathscr{S}'(\mathbf{R}^n)$. Suppose that $u_N \to u$ in $\mathscr{S}'/\mathscr{P}$ and $u_N \to v$ in $\mathscr{S}'$. Then prove that $u - v$ is a polynomial.
[*Hint:* Use Proposition 1.1.3 or directly Proposition 2.4.1 in [156].]

1.1.4. Suppose that Ψ is a Schwartz function whose Fourier transform is supported in an annulus that does not contain the origin and satisfies $\sum_{j \in \mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$. Show that for functions $g \in L^1(\mathbf{R}^n)$ with $\widehat{g} \in L^1(\mathbf{R}^n)$ we have $\sum_{j \in \mathbf{Z}} \Delta_j^\Psi(g) = g$ pointwise everywhere.

1.1.5. Let Θ and Φ be Schwartz functions whose Fourier transforms are compactly supported and let Ψ, Ω be Schwartz functions whose Fourier transforms are supported in annuli that do not contain the origin and satisfy

$$\widehat{\Phi}(\xi)\widehat{\Theta}(\xi) + \sum_{j=1}^\infty \widehat{\Psi}(2^{-j}\xi)\widehat{\Omega}(2^{-j}\xi) = 1$$

for all $\xi \in \mathbf{R}^n$. Then for all $f \in \mathscr{S}'(\mathbf{R}^n)$ we have

$$\Phi * \Theta * f + \sum_{j=1}^\infty \Delta_j^\Psi \Delta_j^\Omega(f) = f$$

where the series converges in $\mathscr{S}'(\mathbf{R}^n)$.

1.1.6. (a) Show that for any multi-index α on $\mathbf{R}^n$ there is a polynomial Q_α of n variables of degree $|\alpha|$ such that for all $\xi \in \mathbf{R}^n$ we have

$$\partial^\alpha(e^{-|\xi|^2}) = Q_\alpha(\xi) e^{-|\xi|^2} .$$

(b) Show that for all multi-indices $|\alpha| \geq 1$ and for each k in $\{0, 1, \dots, |\alpha| - 1\}$ there is a polynomial $P_{\alpha,k}$ of n variables of degree at most $|\alpha|$ such that

$$\partial^\alpha(e^{-|\xi|}) = \sum_{k=0}^{|\alpha|-1} \frac{1}{|\xi|^k} P_{\alpha,k}\left(\frac{\xi_1}{|\xi|}, \dots, \frac{\xi_n}{|\xi|}\right) e^{-|\xi|}$$