where $N = [\frac{n}{p}]+1$, and consequently there is a constant $C_{n,p}$ such that

$$\big|\big\langle f,\varphi\big\rangle\big| \le \mathfrak{N}_N(\varphi)\,C_{n,p}\,\big\|f\big\|_{H^p}\,. \tag{2.1.50}$$

(b) Let $0<p\le 1$, $N=[n/p]+1$, and $p\le r\le\infty$. Then there is a constant $C(p,n,r)$ such that for any $f\in H^p$ and $\varphi\in\mathscr{S}(\mathbf{R}^n)$ we have

$$\big\|\varphi * f\big\|_{L^r} \le C(p,n,r)\mathfrak{N}_N(\varphi)\big\|f\big\|_{H^p}\,. \tag{2.1.51}$$

(c) For any $x_0\in\mathbf{R}^n$, for all $R>0$, and any $\psi\in\mathscr{S}(\mathbf{R}^n)$, we have

$$\big|\big\langle f,\psi\big\rangle\big| \le \mathfrak{N}_N(\psi;x_0,R)\inf_{|z-x_0|\le R}\mathscr{M}_N(f)(z)\,. \tag{2.1.52}$$

Proof. (a) We use that $\langle f,\varphi\rangle = (\widetilde{\varphi}*f)(0)$, where $\widetilde{\varphi}(x)=\varphi(-x)$ and we observe that $\mathfrak{N}_N(\varphi)=\mathfrak{N}_N(\widetilde{\varphi})$. Then (2.1.49) follows from the inequality

$$|(\widetilde{\varphi}*f)(0)| \le \mathfrak{N}_N(\varphi)M_1^*\Big(f;\frac{\widetilde{\varphi}}{\mathfrak{N}_N(\varphi)}\Big)(z) \le \mathfrak{N}_N(\varphi)\mathscr{M}_N(f)(z)$$

for all $|z|<1$, which is valid, since $\widetilde{\varphi}/\mathfrak{N}_N(\varphi)$ lies in $\mathscr{F}_N$. We deduce (2.1.50) as follows:

$$\begin{aligned}\big|\big\langle f,\varphi\big\rangle\big|^p &\le \mathfrak{N}_N(\varphi)^p\inf_{|z|\le 1}\mathscr{M}_N(f)(z)^p\\ &\le \mathfrak{N}_N(\varphi)^p\,\frac{1}{|B(0,1)|}\int_{|z|\le 1}\mathscr{M}_N(f)^p\,dz\\ &\le \mathfrak{N}_N(\varphi)^p\,C_{n,p}^p\,\big\|f\big\|_{H^p}^p\,.\end{aligned}$$

(b) For any fixed $x\in\mathbf{R}^n$ and $t>0$ we have

$$|(\varphi_t*f)(x)| \le \mathfrak{N}_N(\varphi)M_1^*\Big(f;\frac{\varphi}{\mathfrak{N}_N(\varphi)}\Big)(y) \le \mathfrak{N}_N(\varphi)\mathscr{M}_N(f)(y) \tag{2.1.53}$$

for all y satisfying $|y-x|\le t$. Restricting to $t=1$ yields

$$|(\varphi*f)(x)|^p \le \frac{\mathfrak{N}_N(\varphi)^p}{|B(x,1)|}\int_{B(x,1)}\mathscr{M}_N(f)(y)^p\,dy \le \mathfrak{N}_N(\varphi)^pC_{p,n}^p\big\|f\big\|_{H^p}^p\,.$$

This implies that $\|\varphi*f\|_{L^\infty}\le C_{p,n}\mathfrak{N}_N(\varphi)\|f\|_{H^p}$. Choosing $y=x$ and $t=1$ in (2.1.53) and then taking L^p quasi-norms yields a similar estimate for $\|\varphi*f\|_{L^p}$. By interpolation we deduce $\|\varphi*f\|_{L^r}\le C(p,n,r)\mathfrak{N}_N(\varphi)\|f\|_{H^p}$, when $p\le r\le\infty$.

(c) To prove (2.1.52), given a Schwartz function ψ and $R>0$, define $\varphi(y)=\psi(-Ry+x_0)$ so that $\psi(x)=\varphi(\frac{x_0-x}{R})=R^n\varphi_R(x_0-x)$. In view of (2.1.53) we have

$$\big|\big\langle f,\psi\big\rangle\big| = R^n\big|(\varphi_R*f)(x_0)\big| \le R^n\,\mathfrak{N}_N(\varphi)\inf_{|z-x_0|\le R}\mathscr{M}_N(f)(z)\,.$$

But a simple change of variables shows that $R^n\mathfrak{N}(\varphi) = \mathfrak{N}(\psi;x_0,R)$ and this combined with the preceding inequality yields (2.1.52).

□

Proposition 2.1.10. *Let $0 < p \le 1$. Then the following statements are valid:*
(a) Convergence in H^p implies convergence in $\mathscr{S}'$.
(b) If $f_k \in H^p$ satisfy $\sup_{k\in\mathbf{Z}^+}\|f_k\|_{H^p} \le C < \infty$ and $f_k \to f$ in $\mathscr{S}'(\mathbf{R}^n)$ as $k\to\infty$, then $f \in H^p$.
(c) H^p is a complete quasi-normed metrizable space.

Proof. (a) Let f_j, f in $H^p(\mathbf{R}^n)$ and suppose that $f_j \to f$ in $H^p(\mathbf{R}^n)$. Applying (2.1.50) we obtain that for any $\varphi \in \mathscr{S}(\mathbf{R}^n)$ we have $\langle f_j - f, \varphi\rangle \to 0$; hence $f_j \to f$ in $\mathscr{S}'(\mathbf{R}^n)$.

(b) For any $\Phi \in \mathscr{S}(\mathbf{R}^n)$ with integral one and $t>0$ we have $\Phi_t * f_k \to \Phi_t * f$ as $k\to\infty$, since $f_k \to f$ in $\mathscr{S}'(\mathbf{R}^n)$. Thus

$$|\Phi_t * f| = \liminf_{k\to\infty}|\Phi_t * f_k| \le \liminf_{k\to\infty}\sup_{t>0}|\Phi_t * f_k|\,.$$

Taking the supremum over t, we obtain $\sup_{t>0}|\Phi_t * f| \le \liminf_{k\to\infty}\sup_{t>0}|\Phi_t * f_k|$. Then we apply L^p quasi-norms and Fatou's lemma to deduce that $\|M(f;\Phi)\|_{L^p}$ is bounded by a multiple of C; thus, $f \in H^p$.

(c) Suppose $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $H^p(\mathbf{R}^n)$. Then there is a constant C_0 such that $\sup_{j\ge1}\|f_j\|_{H^p} \le C_0$. Using (2.1.50) (with $f_j - f_k$ in place of f) we obtain that for every φ in $\mathscr{S}(\mathbf{R}^n)$ the sequence $\{\langle f_j,\varphi\rangle\}_{j=1}^\infty$ is Cauchy in $\mathbf{C}$ and thus it converges to a complex number $f(\varphi)$. We claim that the mapping $\varphi \mapsto f(\varphi)$ is a tempered distribution. We clearly have

$$|f(\varphi)| = \lim_{k\to\infty}\big|\langle f_k,\varphi\rangle\big| \le C_{n,p}\mathfrak{N}_N(\varphi)C_0\,.$$

But an easy calculation shows that $\mathfrak{N}_N(\varphi)$ is controlled by the finite sum of seminorms $\rho_{\alpha,\beta}(\varphi)$ with $|\alpha|,|\beta| \le N+n+1$. This yields that f lies in $\mathscr{S}'(\mathbf{R}^n)$, in particular f is a bounded distribution, and obviously $f_j \to f$ in $\mathscr{S}'(\mathbf{R}^n)$. Part (b) implies that f is an element of $H^p(\mathbf{R}^n)$.

Next we show that $f_k \to f$ in H^p. Given $\Phi \in \mathscr{S}(\mathbf{R}^n)$ with integral 1, we have for any $t>0$ and any $k\ge1$

$$|(f_k - f)*\Phi_t| = \liminf_{\ell\to\infty}|(f_k - f_\ell)*\Phi_t| \le \liminf_{\ell\to\infty}\sup_{t>0}|(f_k - f_\ell)*\Phi_t|\,.$$

Taking the supremum over $t>0$ on the left and then the L^p quasi-norm and applying Fatou's lemma we deduce that

$$\big\|M(f_k - f;\Phi)\big\|_{L^p} \le \liminf_{\ell\to\infty}\big\|M(f_k - f_\ell;\Phi)\big\|_{L^p}\,.$$

Letting $k\to\infty$ we obtain that

$$\limsup_{k\to\infty}\big\|M(f_k - f;\Phi)\big\|_{L^p} \le \limsup_{k,\ell\to\infty}\big\|M(f_k - f_\ell;\Phi)\big\|_{L^p} = 0;$$