Fact: If $g \in L^\infty(\mathbf{R}^n)$ is continuous in a neighborhood of $x_0 \in \mathbf{R}^n$, then

$$(P_s * g)(x) \to g(x_0)$$

as $(x, s) \to (x_0, 0^+)$.

Proof: Given $\varepsilon > 0$ there is an $s_1 > 0$ such that for $s < s_1$ we have

$$|(P_s * g)(x_0) - g(x_0)| < \frac{\varepsilon}{2}$$

since g is continuous at x_0. Since g is uniformly continuous on a ball $\overline{B(x_0, \delta_0)}$, there is a $\delta \in (0, \frac{\delta_0}{2})$ such that $|g(z) - g(z')| \leq \frac{\varepsilon}{4}$ whenever $|z - x_0| \leq \delta$. For $|y|, |x - x_0| < \delta$ we have $x - y, x_0 - y \in \overline{B(x_0, \delta_0)}$ and

$$\left| \int_{|y|<\delta} P_s(y)[g(x-y) - g(x_0 - y)]\, dy \right| \leq \frac{\varepsilon}{4}$$

since $|(x - y) - (x_0 - y)| < \delta$. Also, choose $s_2 > 0$ such that

$$\left| \int_{|y|\geq\delta} P_s(y)[g(x-y) - g(x_0 - y)]\, dy \right| \leq 2\|g\|_{L^\infty} \frac{c_n\, s}{\delta^{1+\frac{n}{r}}} \leq \frac{\varepsilon}{4}$$

for all $s < s_2$. Then for $|x - x_0| < \delta$ and $s < \min(s_1, s_2)$,

$$|(P_s*g)(x) - g(x_0)| \leq |(P_s*g)(x) - (P_s*g)(x_0)| + |(P_s*g)(x_0) - g(x_0)| < \left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) + \frac{\varepsilon}{2} = \varepsilon.$$