It follows from the definition of $M_a^*(f;\Phi)(z) = \sup_{t>0}\sup_{|w-z|<at}|(\Phi_t * f)(w)|$ that

$$|(\Phi_t * f)(x-y)| \le M_a^*(f;\Phi)(z) \qquad \text{if } z \in B(x-y,at)\,.$$

But the ball $B(x-y,at)$ is contained in the ball $B(x,|y|+at)$; hence it follows that

$$\begin{aligned}
|(\Phi_t * f)(x-y)|^{\frac{n}{b}} &\le \frac{1}{|B(x-y,at)|}\int_{B(x-y,at)} M_a^*(f;\Phi)(z)^{\frac{n}{b}}\,dz \\
&\le \frac{1}{|B(x-y,at)|}\int_{B(x,|y|+at)} M_a^*(f;\Phi)(z)^{\frac{n}{b}}\,dz \\
&\le \left(\frac{|y|+at}{at}\right)^n M\big(M_a^*(f;\Phi)^{\frac{n}{b}}\big)(x) \\
&\le \max(1,a^{-n})\left(\frac{|y|}{t}+1\right)^n M\big(M_a^*(f;\Phi)^{\frac{n}{b}}\big)(x)\,,
\end{aligned}$$

from which we conclude that for all $x \in \mathbf{R}^n$ we have

$$M_b^{**}(f;\Phi)(x) \le \max(1,a^{-b})\left\{M\big(M_a^*(f;\Phi)^{\frac{n}{b}}\big)(x)\right\}^{\frac{b}{n}}.$$

Raising to the power p and using the fact that $p > n/b$ and the boundedness of the Hardy–Littlewood maximal operator M on $L^{pb/n}$, we obtain the required conclusion (2.1.14).

(d) In proving (d) we may replace b by the integer $b_0 = [b]+1$. Let Φ be a Schwartz function with integral equal to 1. Applying Lemma 2.1.5 with $m = b_0$, we write any function φ in $\mathscr{S}(\mathbf{R}^n)$ as

$$\varphi(y) = \int_0^1 (\Theta^{(s)} * \Phi_s)(y)\,ds$$

for some choice of Schwartz functions $\Theta^{(s)}$. Then we have

$$\varphi_t(y) = \int_0^1 ((\Theta^{(s)})_t * \Phi_{ts})(y)\,ds$$

for all $t > 0$. Fix $x \in \mathbf{R}^n$. Then for y in $B(x,t)$ we have

$$\begin{aligned}
|(\varphi_t * f)(y)| \le & \int_0^1\int_{\mathbf{R}^n} |(\Theta^{(s)})_t(z)|\,|(\Phi_{ts} * f)(y-z)|\,dz\,ds \\
&\le \int_0^1\int_{\mathbf{R}^n} |(\Theta^{(s)})_t(z)|\,M_{b_0}^{**}(f;\Phi)(x)\left(\frac{|x-(y-z)|}{st}+1\right)^{b_0} dz\,ds \\
&\le \int_0^1 s^{-b_0}\int_{\mathbf{R}^n} |(\Theta^{(s)})_t(z)|\,M_{b_0}^{**}(f;\Phi)(x)\left(\frac{|x-y|}{t}+\frac{|z|}{t}+1\right)^{b_0} dz\,ds \\
&\le 2^{b_0} M_{b_0}^{**}(f;\Phi)(x)\int_0^1 s^{-b_0}\int_{\mathbf{R}^n} |\Theta^{(s)}(w)|\,\big(|w|+1\big)^{b_0}\,dw\,ds \\
&\le 2^{b_0} M_{b_0}^{**}(f;\Phi)(x)\int_0^1 s^{-b_0} C_0(\Phi,b_0)\,s^{b_0}\,\mathfrak{N}_{b_0}(\varphi)\,ds\,,
\end{aligned}$$

where we applied conclusion (2.1.18) of Lemma 2.1.5. Setting $N = b_0 = [b] + 1$, we obtain for y in $B(x,t)$ and $\varphi \in \mathscr{S}(\mathbf{R}^n)$,

$$|(\varphi_t * f)(y)| \le 2^{b_0} C_0(\Phi, b_0)\, \mathfrak{N}_{b_0}(\varphi)\, M_{b_0}^{**}(f;\Phi)(x)\,.$$

Taking the supremum over all y in $B(x,t)$, over all $t > 0$, and over all φ in $\mathscr{F}_N$, we obtain the pointwise estimate

$$\mathscr{M}_N(f)(x) \le 2^{b_0} C_0(\Phi, b_0)\, M_{b_0}^{**}(f;\Phi)(x)\,, \qquad x \in \mathbf{R}^n,$$

where $N = b_0$. This clearly yields (2.1.15) if we set $C_4 = 2^{b_0} C_0(\Phi, b_0)$.

(e) We fix an $f \in \mathscr{S}'(\mathbf{R}^n)$ that satisfies $\|\mathscr{M}_N(f)\|_{L^p} < \infty$ for some fixed positive integer N. To show that f is a bounded distribution, we fix a Schwartz function φ and we observe that for some positive constant $c = c_\varphi$, we have that $c\,\varphi$ is an element of $\mathscr{F}_N$ and thus $M_1^*(f; c\,\varphi) \le \mathscr{M}_N(f)$. Then

$$\begin{aligned}
c^p\,|(\varphi * f)(x)|^p &\le \inf_{|y-x|\le 1} \sup_{|z-y|\le 1} |(c\,\varphi * f)(z)|^p \\
&\le \inf_{|y-x|\le 1} M_1^*(f; c\,\varphi)(y)^p \\
&\le \frac{1}{v_n} \int_{|y-x|\le 1} M_1^*(f; c\,\varphi)(y)^p\, dy \\
&\le \frac{1}{v_n} \int_{\mathbf{R}^n} M_1^*(f; c\,\varphi)(y)^p\, dy \\
&\le \frac{1}{v_n} \int_{\mathbf{R}^n} \mathscr{M}_N(f)(y)^p\, dy < \infty,
\end{aligned}$$

which implies that $\varphi * f$ is a bounded function. We conclude that f is a bounded distribution. We now proceed to show that f is an element of H^p. We fix a smooth radial nonnegative compactly supported function θ such that

$$\theta(x) = \begin{cases} 1 & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| > 2. \end{cases}$$

We observe that the identity

$$\begin{aligned}
P(x) &= P(x)\theta(x) + \sum_{k=1}^{\infty} \big(\theta(2^{-k}x)P(x) - \theta(2^{-(k-1)}x)P(x)\big) \\
&= P(x)\theta(x) + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} \left(\frac{\theta(\,\cdot\,) - \theta(2(\,\cdot\,))}{(2^{-2k} + |\cdot|^2)^{\frac{n+1}{2}}} \right)_{2^k}(x)
\end{aligned}$$

is valid for all $x \in \mathbf{R}^n$. We set

$$\Phi^{(k)}(x) = \big(\theta(x) - \theta(2x)\big) \frac{1}{(2^{-2k} + |x|^2)^{\frac{n+1}{2}}}\,,$$

and we claim that for all bounded tempered distributions f and for all $t > 0$ we have

$$P_t * f = (\theta P)_t * f + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} (\Phi^{(k)})_{2^k t} * f \,, \tag{2.1.38}$$

where the series converges in $\mathscr{S}'(\mathbf{R}^n)$; see Exercise 2.1.5.

Assuming (2.1.38), we claim that for some fixed constant $c_0 = c_0(n,N)$, the functions $c_0\,\theta\,P$ and $c_0 \Phi^{(k)}$ lie in $\mathscr{F}_N$ uniformly in $k = 1,2,3,\dots$.

To verify this assertion for $|\alpha| \le N+1$, we apply Leibniz's rule to write

$$\begin{aligned}
\left| \partial^\alpha \left[\frac{\theta(x) - \theta(2x)}{(2^{-2k} + |x|^2)^{\frac{n+1}{2}}} \right] \right| &= \left| \sum_{\beta \le \alpha} c_{\alpha,\beta} \partial_x^{\alpha-\beta} (\theta(x) - \theta(2x)) \partial_x^\beta \left(\frac{1}{(2^{-2k} + |x|^2)^{\frac{n+1}{2}}} \right) \right| \\
&\le \sum_{\beta \le \alpha} |c'_{\alpha,\beta}| \chi_{\frac{1}{2} \le |x| \le 2} \frac{K_\beta}{(2^{-2k} + |x|^2)^{\frac{n+1}{2} + \frac{|\beta|}{2}}} \,,
\end{aligned}$$

where

$$K_\beta = \sup_{\substack{m,\gamma \\ m+|\gamma|=|\beta|}} \sup_{\substack{t,x \\ t^2+|x|^2}} \left| \frac{\partial^m}{\partial t^m} \frac{\partial^\gamma}{\partial x^\gamma} \frac{1}{(t^2 + |x|^2)^{\frac{n+1}{2}}} \right| ,$$

and this estimate follows from the fact that the function $(t^2 + |x|^2)^{-\frac{n+1}{2}}$ is homogeneous of degree $-n-1$ on $\mathbf{R}^{n+1}$ and smooth on the sphere $\mathbf{S}^n$. These estimates are uniform in $k = 0,1,2,\dots$ and thus $\mathfrak{N}_N(\theta P) + \mathfrak{N}_N(\Phi^{(k)}) \le 1/c_0(n,N)$ for all some constant $c_0 = c_0(n,N)$ for all $k = 0,1,2,\dots$.

Then we obtain

$$\begin{aligned}
\sup_{t>0} |P_t * f| &\le \sup_{t>0} |(\theta P)_t * f| + \frac{1}{c_0} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} \sup_{t>0} \left| (c_0 \Phi^{(k)})_{2^k t} * f \right| \\
&\le C_5(n,N) \mathscr{M}_N(f) \,,
\end{aligned}$$

which proves the required conclusion (2.1.16).

We observe that the last estimate also yields the stronger estimate

$$M_1^*(f;P)(x) = \sup_{t>0} \sup_{\substack{y \in \mathbf{R}^n \\ |y-x| \le t}} |(P_t * f)(y)| \le C_5(n,N) \mathscr{M}_N(f)(x) \,. \tag{2.1.39}$$

It follows that the quasi-norm $\|M_1^*(f;P)\|_{L^p(\mathbf{R}^n)}$ is also equivalent to $\|f\|_{H^p}$. $\square$

Remark 2.1.6. To simplify the understanding of the equivalences just proved, a first-time reader may wish to define the H^p quasi-norm of a distribution f as

$$\big\|f\big\|_{H^p} = \big\|M_1^*(f;P)\big\|_{L^p}$$

and then study only the implications (a) $\Longrightarrow$ (c), (c) $\Longrightarrow$ (d), (d) $\Longrightarrow$ (e), and (e) $\Longrightarrow$ (a) in the proof of Theorem 2.1.4. In this way one avoids passing through the statement in part (b). For many applications, the identification of $\|f\|_{H^p}$ with $\|M_1^*(f;\Phi)\|_{L^p}$ for some Schwartz function Φ (with nonvanishing integral) suffices.

We also remark that the proof of Theorem 2.1.4 yields

$$\big\|f\big\|_{H^p(\mathbf{R}^n)} \approx \big\|\mathscr{M}_N(f)\big\|_{L^p(\mathbf{R}^n)},$$

where $N = [\frac{n}{p}]+1$.

2.1.3 Consequences of the Characterizations of Hardy Spaces

In this subsection we look at a few consequences of Theorem 2.1.4. In many applications we need to be working with dense subspaces of H^p. It turns out that both $H^p \cap L^2$ and $H^p \cap L^1$ are dense in H^p.

Proposition 2.1.7. *Let $0 < p \le 1$ and let r satisfy $p \le r \le \infty$. Then $L^r \cap H^p$ is dense in H^p. In particular, $H^p \cap L^2$ and $H^p \cap L^1$ are dense in H^p.*

Proof. Let f be a distribution in $H^p(\mathbf{R}^n)$. Recall the Poisson kernel $P(x)$ and set $N = [\frac{n}{p}]+1$. For any fixed $x \in \mathbf{R}^n$ and $t > 0$ we have

$$|(P_t * f)(x)| \le M_1^*(f;P)(y) \le C\mathscr{M}_N(f)(y) \tag{2.1.40}$$

for any $|y-x| \le t$. Indeed, the first estimate in (2.1.40) follows from the definition of $M_1^*(f;P)$, and the second estimate by (2.1.39). Raising (2.1.40) to the power p and averaging over the ball $B(x,t)$, we obtain

$$|(P_t * f)(x)|^p \le \frac{C^p}{v_n t^n}\int_{B(x,t)} \mathscr{M}_N(f)(y)^p\,dy \le \frac{C_1^p}{t^n}\big\|f\big\|_{H^p}^p. \tag{2.1.41}$$

It follows that the function $P_t * f$ is in $L^\infty(\mathbf{R}^n)$ with norm at most a constant multiple of $t^{-n/p}\big\|f\big\|_{H^p}$. Moreover, this function is also in $L^p(\mathbf{R}^n)$, since it is controlled by $M(f;P)$. Therefore, the functions $P_t * f$ lie in $L^r(\mathbf{R}^n)$ for all r with $p \le r \le \infty$. It remains to show that $P_t * f$ also lie in H^p and that $P_t * f \to f$ in H^p as $t \to 0$.

To see that $P_t * f$ lies in H^p, we use the semigroup formula $P_t * P_s = P_{t+s}$ for the Poisson kernel, which is a consequence of the fact that $\widehat{P_t}(\xi) = e^{-2\pi t|\xi|}$ by applying the Fourier transform. Therefore, for any $t > 0$ we have

$$\sup_{s>0}|P_s * P_t * f| = \sup_{s>0}|P_{s+t} * f| \le \sup_{s>0}|P_s * f|,$$

which implies that

$$\big\|P_t * f\big\|_{H^p} \le \big\|f\big\|_{H^p}$$

for all $t > 0$. We now need to show that $P_t * f \to f$ in H^p as $t \to 0$. This will be a consequence of the Lebesgue dominated convergence theorem once we know that

$$\sup_{s>0}|P_s * P_t * f - P_s * f| \le 2\sup_{s>0}|P_s * f| \in L^p(\mathbf{R}^n), \tag{2.1.42}$$

which will be a consequence of the semigroup formula $P_{s+t} = P_s * P_t$ and the a.e. convergence property (2.1.43) of the subsequent theorem. □

Theorem: (A. Calderón) *Let* $0 < p < \infty$. *The for any* $f \in H^p(\mathbf{R}^n)$, *the harmonic function* $(x,s) \mapsto (P_s * f)(x)$ *on* $\mathbf{R}^{n+1}_+$ *satisfies for almost all* $x \in \mathbf{R}^n$

$$\sup_{s>0} |(P_t * P_s * f - P_s * f)(x)| \to 0 \quad \text{as} \quad t \to 0. \tag{2.1.43}$$

Proof. To show that $(x,s) \mapsto (P_s * f)(x)$ is harmonic on $\mathbf{R}^{n+1}_+$ we fix a function η_0 whose Fourier transform is supported in a ball centered at 0 and we let $\eta_\infty = 1 - \eta_0$. Then $P_s * f = P_s * (\eta_0 * f) + (P_s * \eta_\infty) * f$. The convolutions are well defined as $P_s * \eta_\infty$ is a Schwartz function and $\eta_0 * f$ is a bounded function, since f is a bounded distribution. Let Δ be the Laplacian in both $x \in \mathbf{R}^n$ and $s > 0$. Then we have $\Delta(P_s * f) = \Delta(P_s) * (\eta_0 * f) + \Delta(P_s * \eta_\infty) * f$ but $\Delta(P_s) = 0$ and also $\Delta(P_s * \eta_\infty) = 0$, which easily follows by applying Δ to $(P_s * \eta_\infty)(x) = \int_{\mathbf{R}^n} e^{-s2\pi|\xi|} \widehat{\eta_\infty}(\xi) e^{2\pi i x\cdot\xi} d\xi$.

We begin the proof of (2.1.43) by fixing $\varepsilon > 0$. In view of (2.1.41), we have

$$\sup_{x\in\mathbf{R}^n} \sup_{s>M} |(P_s * P_t * f - P_s * f)(x)| \le C' M^{-n/p}$$

and we pick M such that $C' M^{-n/p} < \varepsilon/2$. It will suffice to show that there is a subset E of $\mathbf{R}^n$ of Lebesgue measure zero such that for all $x \in \mathbf{R}^n \setminus E$ we have

$$\sup_{0<s\le M} |(P_{t+s} * f)(x) - (P_s * f)(x)| < \varepsilon/2, \tag{2.1.44}$$

provided t is sufficiently close to zero (depending on x). We claim that there is set of measure zero E such that for all $x \in \mathbf{R}^n \setminus E$

$$(P_s * f)(x) \quad \text{converges as} \quad s \to 0^+. \tag{2.1.45}$$

Assuming (2.1.45) we obtain (2.1.44) via the following argument. For $x \in \mathbf{R}^n \setminus E$ the function $s \mapsto (P_s * f)(x)$ is continuous on $(0,M]$ and has a limit as $s \to 0$, hence it has a uniformly continuous extension on $[0,M]$. Then for $x \in \mathbf{R}^n \setminus E$ and for the given $\varepsilon > 0$ there is a $t_x > 0$ such that (2.1.44) holds for all t satisfying $0 < t < t_x$.

We therefore focus on (2.1.45). Since $\|\mathscr{M}_N(f)\|_{L^p} < \infty$, there is a null set E' consisting of all $x \in \mathbf{R}^n$ for which $\mathscr{M}_N(f)(x) = \infty$. We begin by observing that, as a consequence of (2.1.39), for every $x \in \mathbf{R}^n \setminus E'$ there is a constant $C_x < \infty$ such that

$$\sup_{t>0} \sup_{\substack{y\in\mathbf{R}^n \\ |y-x|\le t}} |(P_t * f)(y)| \le C_x,$$

in other words, the harmonic function $(y,t) \mapsto u(y,t) = (P_t * f)(y)$ is bounded on the cone $\Gamma_x = \{(z,s) \in \mathbf{R}^n \times \mathbf{R}^+ : \ |x-z| < s\}$. Letting $U_m = \{x \in \mathbf{R}^n : \ |u| \le m \text{ on } \Gamma_x\}$ for $m \in \mathbf{Z}^+$ we have $\bigcup_{m=1}^\infty U_m = \mathbf{R}^n \setminus E'$, and so it suffices to prove (2.1.45) for almost all x in a given U_m. Moreover, as $\mathbf{R}^n$ is a countable union of cubes, we may also assume that the given U_m is intersected with a fixed cube Q. Having made all these assumptions, for $j \in \mathbf{Z}^+$ we define sets

$$G_j = \bigcup_{y\in U_m\cap Q} B\big(y, \tfrac{1}{j}\big) = \mathbf{R}^n \cap \bigcup_{y\in U_m\cap Q} (\Gamma_y - \tfrac{1}{j}),$$

where for a given cone Γ_y and $\delta > 0$, $\Gamma_y - \delta$ denotes the set $\{(z, s-\delta) : (z,s) \in \Gamma_y\}$, i.e., the vertical downwards translation of Γ_y by the quantity δ. Notice that all G_j $(j = 1,2,\dots)$ are open sets contained in a fixed compact set Q'.

Consider the sequence of functions $F_j(x) = \chi_{G_j}(x)(P_{1/j} * f)(x)$, $j = 1,2,\dots$. We claim that these functions bounded by $m\chi_{Q'}$ uniformly in j. Indeed, given $x \in G_j$ we have $x \in \Gamma_y - \frac{1}{j}$ for some $y \in U_m \cap Q$, thus $(x, \frac{1}{j}) \in \Gamma_y$ and $|(P_{1/j} * f)(x)| \le m$. Thus the functions F_j lie in a multiple of the unit ball of $L^2(\mathbf{R}^n)$ and by the Banach-Alaoglou theorem there is a subsequence $\{j_k\}_{k=1}^\infty$ of $\mathbf{Z}^+$ and a function $F \in L^2(\mathbf{R}^n)$ such that $F_{j_k} \to F$ weakly in L^2. In particular, this implies that $P_s * F_{j_k} \to P_s * F \quad$ as $k \to \infty$. For any $s > 0$ and $x \in \mathbf{T}^n$ we write $(P_s * f)(x) = (P_s * f)(x) - (P_s * F)(x) + (P_s * F)(x)$ and to prove (2.1.45), we will use that[1]

$$(P_s * F)(x) - F(x) \to 0 \quad \text{for almost all } x \in \mathbf{R}^n, \tag{2.1.46}$$

[which also holds for functions $F_0 \in L^\infty(\mathbf{R}^n)$], and

$$(P_s * f)(x) - (P_s * F)(x) \to 0 \quad \text{for almost all } x \in \mathbf{R}^n. \tag{2.1.47}$$

Now consider the harmonic function on $\mathbf{R}^{n+1}_+$ given by

$$w(x,t) = 2mt + c \int_{\mathbf{R}^n \setminus (U_m \cap Q)} P_t(x-y)\, dy$$

for some $c > 0$ to be determined. Define an open set

$$\Omega = \bigcup_{y \in U_m \cap Q} \Gamma_y \cap \{(z,t) \in \mathbf{R}^{n+1}_+ : t < 1\}.$$

We claim that

$$w(x,t) \ge 2m \quad \text{ on } \partial\Omega \setminus (U_m \cap Q). \tag{2.1.48}$$

and that the limit of $w(x,t)$ exists as $t \to 0^+$ for almost all $x \in U_m \cap Q$; to see this last assertion we apply (2.1.46) to the bounded function $F_0 = \chi_{\mathbf{R}^n \setminus (U_m \cap Q)}$.

We now prove (2.1.48). This assertion is obvious on the top part of the boundary of Ω. Let $(x_0, t_0) \in \partial\Omega$ with $0 < t_0 < 1$. For (x_0,t_0) in $\mathbf{R}^{n+1}_+$ consider the inverted cone $\Gamma^{(x_0,t_0)} = \{(y,s) \in \mathbf{R}^{n+1}_+ : |y - x_0| < t_0 - s\}$ which satisfies $\Gamma^{(x_0,t_0)} \cap \mathbf{R}^n = B(x_0,t_0)$. We claim that $B(x_0,t_0) \cap U_m \cap Q = \emptyset$; indeed, if $B(x_0,t_0) \cap U_m \cap Q$ contained a point y_0, then x_0 would lie in $\Gamma_{y_0} \cap \{(z,t) \in \mathbf{R}^{n+1}_+ : t < 1\}$ which is impossible since (x_0,t_0) lies in the boundary of Ω. Then

$$w(x_0,t_0) \ge \int_{\mathbf{R}^n \setminus (U_m \cap Q)} c P_{t_0}(x_0 - y) dy \ge \int_{|x_0 - y| \le t_0} c P_{t_0}(x_0 - y) dy = c \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_0^1 \frac{\omega_{n-1} r^{n-1} dr}{(1+r^2)^{\frac{n+1}{2}}}$$

and this can be made to be larger than $2m$ by choosing $c = c(n,m)$ suitably. Thus (2.1.48) also holds on the non horizontal part of the boundary of Ω.

We finally focus on the proof of (2.1.47). We observe that $(P_s * f)(x) - (P_s * F)(x)$ is equal to the pointwise limit of $(P_s * P_{1/j_k} * f)(x) - (P_s * F_{j_k})(x)$ as $k \to \infty$ for all $x \in \mathbf{R}^n$ and $s > 0$. We claim that for every j_k we have

[1] For a proof see the book's website: http://faculty.missouri.edu/~grafakosl/FourierAnalysis.html

(i) $|(P_s*P_{1/j_k}*f)(x)-(P_s*F_{j_k})(x)|\le 2m$ for all $(x,s)\in\Omega$,
(ii) $|(P_s*P_{1/j_k}*f)(x)-(P_s*F_{j_k})(x)|\to 0$ as $\Omega\ni(x,s)\to(x_0,0^+)$, where $x_0\in\overline{G_{2j_k}}$.

To prove (i) note that for a given $(x,s)\in\Omega$, there is a $y\in U_m\cap Q$ such that $(x,s)\in\Gamma_y$. Then $(x,s+1/j_k)\in\Gamma_y$, hence $|(P_s*P_{1/j_k}*f)(x)|=|(P_{s+1/j_k}*f)(x)|\le m$ as well. Moreover, $\|F_{j_k}\|_{L^\infty}\le m$, thus $|(P_s*F_{j_k})(x)|\le m$ for any $x\in\mathbf{R}^n$. For (ii) we use the following[2]: if $g\in L^\infty(\mathbf{R}^n)$ is continuous in a neighborhood of $x_0\in\mathbf{R}^n$, then $(P_s*g)(x)\to g(x_0)$ as $(x,s)\to(x_0,0^+)$. Then we obtain (ii) by applying this fact to $g=P_{1/j_k}*f-\chi_{G_{j_k}}(P_{1/j_k}*f)$, which is continuous in a neighborhood of a point x_0 in $\overline{G_{2j_k}}$, lies in L^∞ in view of (2.1.41) with $t=1/j_k$, and satisfies $g(x_0)=0$.

We finally prove (2.1.47). Let $w_k(x,t)=(P_t*P_{1/j_k}*f)(x)-(P_t*F_{j_k})(x)$ for $k=1,2,\dots$ and consider the harmonic functions $w(x,t)\pm w_k(x,t)$ on Ω. We have that $w(x,t)\pm w_k(x,t)\ge 0$ on $\partial\Omega\setminus(U_m\cap Q)$ and moreover,

$$\liminf_{\Omega\ni(x,t)\to(x_0,0)}\big(w(x,t)\pm w_k(x,t)\big)\ge\liminf_{\Omega\ni(x,t)\to(x_0,0)}w(x,t)+\liminf_{\Omega\ni(x,t)\to(x_0,0)}(\pm w_k(x,t))\ge 0$$

as the first term on the right is nonnegative and the second one is zero in view of (ii). From this we will deduce that $w(x,t)\pm w_k(x,t)\ge 0$ for all $(x,t)\in\Omega$ and all k. If this were not the case, then $w-w_k$ would take a negative value in Ω and by the minimum principle for harmonic functions it should attain its (negative) minimum at some point in $U_m\cap Q$. Then for some $\delta>0$ and for each $k=1,2,\dots$ there would exist (x_l^k,t_l^k) in Ω such that $w(x_l^k,t_l^k)\pm w_k(x_l^k,t_l^k)<-\delta$ and (x_l^k,t_l^k) would converge to $(x_0,0^+)$ as $l\to\infty$ for some point x_0 in $U_m\cap Q$. But this x_0 lies in $\overline{G_{2j_k}}$ for some j_k contradicting (ii). We now showed that $|w_k|\le w$ on Ω for all k and thus $|\lim_{k\to\infty}w_k|\le w$ on Ω. But since $\lim_{s\to0^+}w(x,s)=0$ for almost all $x\in U_m\cap Q$, the same assertion is valid for $\lim_{k\to\infty}w_k=P_s*f-P_s*F$. This proves (2.1.47). □

Corollary 2.1.8. *For any two Schwartz functions Φ and Θ with nonvanishing integral we have*

$$\big\|\sup_{t>0}|\Theta_t*f|\big\|_{L^p}\approx\big\|\sup_{t>0}|\Phi_t*f|\big\|_{L^p}\approx\big\|f\big\|_{H^p}$$

for all $f\in\mathscr{S}'(\mathbf{R}^n)$, with constants depending only on n,p,Φ, and Θ.

Proof. See the discussion after Theorem 2.1.4. □

Next we define a *norm* on Schwartz functions relevant in the theory of Hardy spaces:

$$\mathfrak{N}_N(\varphi;x_0,R)=\int_{\mathbf{R}^n}\Big(1+\Big|\frac{x-x_0}{R}\Big|\Big)^N\sum_{|\alpha|\le N+1}R^{|\alpha|}|\partial^\alpha\varphi(x)|\,dx\,.$$

Note that $\mathfrak{N}_N(\varphi;0,1)=\mathfrak{N}_N(\varphi)$.

Corollary 2.1.9. *(a) For any $0<p\le1$, every $f\in H^p(\mathbf{R}^n)$, and any $\varphi\in\mathscr{S}(\mathbf{R}^n)$, we have*

$$\big|\langle f,\varphi\rangle\big|\le\mathfrak{N}_N(\varphi)\inf_{|z|\le1}\mathscr{M}_N(f)(z)\,,\tag{2.1.49}$$

[2] For a proof see the book's website: http://faculty.missouri.edu/∼grafakosl/FourierAnalysis.html