(c) Assertion (1.1.6) follows from (1.1.5) by duality. To prove (1.1.5), we use the Fourier transform. We have that if φ_N, φ lie in $\mathscr{S}_0$, then $\varphi_N \to \varphi$ in $\mathscr{S}_0$ if and only if $\varphi_N \to \varphi$ in $\mathscr{S}$ which happens if and only if $\widehat{\varphi_N} \to \widehat{\varphi}$ in $\mathscr{S}$. Thus to show that $\sum_{|j|<N} \Delta_j^\Psi(\varphi) \to \varphi$, it suffices to show that $\sum_{|j|\geq N} \widehat{\varphi}(\xi)\widehat{\Psi}(2^{-j}\xi) \to 0$ in $\mathscr{S}(\mathbf{R}^n)$. But $\partial_\xi^\beta\big(\xi^\alpha\widehat{\varphi}(\xi)\sum_{j\geq N}\widehat{\Psi}(2^{-j}\xi)\big)$ is supported in $|\xi| \geq c\,2^N$, for some constant $c>0$, and decays rapidly at infinity, so

$$\sup_{\xi\in\mathbf{R}^n} |\partial_\xi^\beta\big(\xi^\alpha\widehat{\varphi}(\xi)\sum_{j\geq N}\widehat{\Psi}(2^{-j}\xi)\big)| \to 0$$

as $N\to\infty$. Also, $\partial_\xi^\beta\big(\xi^\alpha\widehat{\varphi}(\xi)\sum_{j\leq -N}\widehat{\Psi}(2^{-j}\xi)\big)$ is supported in $|\xi| \leq c'2^{-N}$ and vanishes at zero to infinite order; thus, it satisfies

$$\big|\partial_\xi^\beta\big(\xi^\alpha\widehat{\varphi}(\xi)\big[\chi_{\{0\}}(\xi)+\sum_{j\leq -N}\widehat{\Psi}(2^{-j}\xi)\big]\big)\big| \leq c_{\alpha,\beta,\varphi,\Psi} \sup_{|\xi|\leq 2^{-N+1}} |\xi|\,,$$

which tends to zero as $N\to\infty$. □

Corollary 1.1.7. ***(Calderón reproducing formula)*** *Let Ψ,Ω be Schwartz functions whose Fourier transforms are supported in annuli that do not contain the origin and satisfy*

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi)\widehat{\Omega}(2^{-j}\xi) = 1$$

for all $\xi\neq 0$. Then for all $f\in\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ we have

$$\sum_{j\in\mathbf{Z}} \Psi_{2^{-j}} * \Omega_{2^{-j}} * f = \sum_{j\in\mathbf{Z}} \Delta_j^\Psi\Delta_j^\Omega(f) = f\,, \tag{1.1.8}$$

where the convergence is in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$.

Proof. The assertion is contained in the conclusion of Proposition 1.1.6(c) with $\Psi*\Omega$ in place of Ψ. □

Exercises

1.1.1. Given multi-indices α,β, show that there are constants C,C' such that

$$\rho_{\alpha,\beta}(\varphi) \leq C \sum_{|\gamma|\leq|\alpha|}\sum_{|\delta|\leq|\beta|} \rho'_{\gamma,\delta}(\varphi)\,,$$

$$\rho'_{\alpha,\beta}(\varphi) \leq C' \sum_{|\gamma|\leq|\alpha|}\sum_{|\delta|\leq|\beta|} \rho_{\gamma,\delta}(\varphi)\,.$$

for all Schwartz functions φ.
$\big[$*Hint:* The first inequality follows by Leibniz's rule. Conversely, to express $\xi^\alpha\partial^\beta\varphi$ in terms of linear combinations of $\partial^\beta(\xi^\gamma\varphi(\xi))$, proceed by induction on $|\alpha|$, using