for all $\xi \in \mathbf{R}^n$. Then for all $\varphi \in \mathscr{S}(\mathbf{R}^n)$ we have

$$S_0^{\Phi}(\varphi) + \sum_{j=1}^{N} \Delta_j^{\Psi}(\varphi) \to \varphi \tag{1.1.3}$$

in $\mathscr{S}(\mathbf{R}^n)$ as $N \to \infty$. Also, for all $f \in \mathscr{S}'(\mathbf{R}^n)$,

$$S_0^{\Phi}(f) + \sum_{j=1}^{N} \Delta_j^{\Psi}(f) \to f \tag{1.1.4}$$

as $N \to \infty$ in the topology of $\mathscr{S}'(\mathbf{R}^n)$.
(c) Let Ψ be a Schwartz function whose Fourier transform is supported in an annulus that does not contain the origin and satisfies

$$\sum_{j \in \mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$$

for all $\xi \neq 0$. Then for all φ in $\mathscr{S}_0(\mathbf{R}^n)$ we have

$$\sum_{|j|<N} \Delta_j^{\Psi}(\varphi) \to \varphi \tag{1.1.5}$$

in $\mathscr{S}_0(\mathbf{R}^n)$ as $N \to \infty$. Also for all f in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ we have that

$$\sum_{|j|<N} \Delta_j^{\Psi}(f) \to f \tag{1.1.6}$$

in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ as $N \to \infty$.

Proof. (a) Let $\widetilde{\Phi}(x) = \Phi(-x)$. We observe that for any $f \in \mathscr{S}'(\mathbf{R}^n)$ and $\varphi \in \mathscr{S}(\mathbf{R}^n)$ we have

$$\left\langle S_N^{\Phi}(f), \varphi \right\rangle = \left\langle f, S_N^{\widetilde{\Phi}}(\varphi) \right\rangle .$$

In view of this, (1.1.2) follows from (1.1.1) via duality, since $\widetilde{\Phi}$ and Φ have the same properties. To prove (1.1.1), we fix a function φ in $\mathscr{S}$. It is equivalent to show that $(S_N^{\Phi}(\varphi))\widehat{\ } \to \widehat{\varphi}$ in $\mathscr{S}(\mathbf{R}^n)$. Fix multi-indices α, β. It will suffice to show that

$$\rho'_{\alpha,\beta}((S_N^{\Phi}(\varphi))\widehat{\ } - \widehat{\varphi}) = \sup_{\xi \in \mathbf{R}^n} \left| \partial_\xi^\beta \left[(1 - \widehat{\Phi}(2^{-N}\xi))\widehat{\varphi}(\xi)\xi^\alpha \right] \right| \to 0 \tag{1.1.7}$$

as $N \to \infty$. Since $\widehat{\Phi}$ is equal to 1 on the unit ball, it follows that the supremum in (1.1.7) is over the set $|\xi| \geq 2^N$. By Leibniz's rule, the ∂^β derivative in the preceding expression is equal to a sum of ∂^γ derivatives falling on $(1 - \widehat{\Phi}(2^{-N}\xi))$ times $\partial^{\beta-\gamma}$ derivatives falling on $\widehat{\varphi}(\xi)\xi^\alpha$, where $\gamma \leq \beta$. If $\gamma \neq 0$, then then a factor of 2^{-N} appears from the differentiation in γ. If $\gamma = 0$, then then the conclusion follows in view of the rapid decay of $\partial^\beta(\widehat{\varphi}(\xi)\xi^\alpha)$ on the set $|\xi| \geq 2^N$.

The proof of (b) follows in the same way as the proof of (a) with the function $\Phi(\xi) + \sum_{j=1}^N \widehat{\Psi}(2^{-j}\xi)$ in place of $\widehat{\Phi}(2^{-N}\xi)$, which has similar support properties.