In particular, we have

$$\big|(\Phi_{2^{-\mu}} * \Psi_{2^{-\nu}})(x)\big| \le C'_{M,N,L,n} A\, B\, \frac{2^{\mu n} 2^{-(\nu-\mu)L}}{(1+2^{\mu}|x|)^M}$$

Let $\Phi_t(x) = t^{-n}\Phi(t^{-1}x)$ and $\Psi_s(x) = s^{-n}\Psi(s^{-1}x)$ for $t,s>0$. Set $2^{-\mu} = t$ and $2^{-\nu} = s$. The assumption $\nu \ge \mu$ can be equivalently stated as $s \le t$.

The preceding inequalities can also be written equivalently as

$$\left|\int_{\mathbf{R}^n} \Phi_t(x-a)\Psi_s(x-b)\,dx\right| \le C'_{M,N,L,n} A\, B\, \frac{t^{-n}\left(\frac{s}{t}\right)^L}{(1+t^{-1}|a-b|)^M}\,.$$

and

$$\big|(\Phi_t * \Psi_s)(x)\big| \le C'_{M,N,L,n} A\, B\, \frac{t^{-n}\left(\frac{s}{t}\right)^L}{(1+t^{-1}|x|)^M}$$

for all $x \in \mathbf{R}^n$, when $s \le t$.

These results are easy consequences of the inequality in Appendix B.2. If Ψ has no cancellation (i.e., $L=0$), then the estimate reduces to that in Appendix B.1.

B.4 Both Functions have Cancellation: An Example

Let $L \in \mathbf{Z}^+$, $A,B,N>0$ and $\mu,\nu \in \mathbf{R}$. Suppose that $N > L+n$. Let Ω, Ψ be $\mathscr{C}^L$ functions on $\mathbf{R}^n$ such that

$$A = \sup_{|\gamma|\le L}\ \sup_{x\in\mathbf{R}^n} |\partial^\gamma \Omega(x)|(1+|x|)^N < \infty$$

$$B = \sup_{|\gamma|\le L}\ \sup_{x\in\mathbf{R}^n} |\partial^\gamma \Psi(x)|(1+|x|)^N < \infty$$

and moreover, for all multi-indices β with $|\beta| \le L-1$ we have

$$\int_{\mathbf{R}^n} \Omega(x)\, x^\beta\, dx = \int_{\mathbf{R}^n} \Psi(x)\, x^\beta\, dx = 0\,.$$

Then given $M>0$ satisfying $M < N-L-n$ there is a constant $C''_{N,M,L,n}$ such that for all $x,a,b \in \mathbf{R}^n$ we have

$$\left|\int_{\mathbf{R}^n} \Omega_{2^{-\mu}}(x-a)\Psi_{2^{-\nu}}(x-b)\,dx\right| \le C''_{N,M,L,n} A\, B\, \frac{\min(2^{\mu n}, 2^{\nu n})2^{-|\nu-\mu|L}}{(1+\min(2^{\mu},2^{\nu})|a-b|)^M}\,.$$