$$\frac{2^{\mu n}}{(1+2^{\mu}|x-a|)^M} \le \frac{2^{\mu n}}{(2^{\mu}\frac{1}{2}|a-b|)^M} \le \frac{4^M 2^{(\nu-\mu)(M-n)}2^{\nu n}}{(1+2^{\nu}|a-b|)^M}\,.$$

The claimed estimate follows.

B.2 One Function has Cancellation

Fix $a,b \in \mathbf{R}^n$, $M \ge 0$, $\mu,\nu \in \mathbf{R}$, and $L \in \mathbf{Z}^+$. Assume that $\nu \ge \mu$ and that $N > L+M+n$.

Given a function Ψ on $\mathbf{R}^n$ and another function $\Phi \in \mathscr{C}^L(\mathbf{R}^n)$ consider the quantities

$$K_{\mu,a}^{M,L}(\Phi) = \sup_{|\beta|=L}\ \sup_{x\in\mathbf{R}^n}(1+2^{\mu}|x-a|)^M|\partial^{\beta}\Phi(x)|\,,$$
$$K_{\nu,b}^{N}(\Psi) = \sup_{x\in\mathbf{R}^n}(1+2^{\nu}|x-b|)^N|\Psi(x)|$$

and assume they are both finite. Suppose, moreover, that

$$\int_{\mathbf{R}^n}\Psi(x)x^{\beta}\,dx = 0 \qquad \text{for all } |\beta| \le L-1.$$

Then there is a constant $C_{M,N,L,n}$ such that

$$\left|\int_{\mathbf{R}^n}\Phi(x)\Psi(x)\,dx\right| \le C_{M,N,L,n}\,K_{\mu,a}^{M,L}(\Phi)\,K_{\nu,b}^{N}(\Psi)\,\frac{2^{-\nu L-\nu n}}{(1+2^{\mu}|a-b|)^M}\,.$$

To prove this claim, we first assume that Φ is real-valued in order to use Langrange's mean value form for the remainder in Taylor's theorem (Appendix I in [156]). For complex-valued Φ we work with its real and imaginary parts separately. We subtract the Taylor polynomial of order $L-1$ of Φ at the point a from the function Φ using the cancellation of Ψ. Then we write

$$\begin{aligned}
&\left|\int_{\mathbf{R}^n}\Phi(x)\Psi(x)\,dx\right| \\
&= \left|\int_{\mathbf{R}^n}\left[\Phi(x) - \sum_{|\gamma|\le L-1}\frac{\partial^{\gamma}\Phi(b)}{\gamma!}(x-b)^{\gamma}\right]\Psi(x)\,dx\right| \\
&= \left|\int_{\mathbf{R}^n}\sum_{|\beta|=L}\frac{\partial^{\beta}\Phi(\xi_{b,x})}{\beta!}(x-b)^{\beta}\,\Psi(x)\,dx\right| \\
&\le K_{\mu,a}^{M,L}(\Phi)\,K_{\nu,b}^{N}(\Psi)\sum_{|\beta|=L}\frac{1}{\beta!}\int_{\mathbf{R}^n}\frac{|x-b|^L}{(1+2^{\mu}|\xi_{b,x}-a|)^M}\frac{1}{(1+2^{\nu}|x-b|)^N}\,dx \\
&\le K_{\mu,a}^{M,L}(\Phi)\,K_{\nu,b}^{N}(\Psi)\sum_{|\beta|=L}\frac{1}{\beta!}\int_{\mathbf{R}^n}\frac{2^{-\nu L}}{(1+2^{\mu}|\xi_{b,x}-a|)^M}\frac{1}{(1+2^{\nu}|x-b|)^{N-L}}\,dx
\end{aligned}$$

where $\xi_{b,x}$ lies on the open segment joining b to x.