7.5.3. (a) Let f_j be in $\mathscr{S}(\mathbf{R}^n)$. Show that for an m-linear multiplier operator T_σ we have that $T_\sigma(f_1,\dots,f_m)\widehat{\ }(\xi)$ is equal to

$$\int_{(\mathbf{R}^n)^{m-1}} \sigma\Big(\xi_1,\dots,\xi_{m-1},\xi-\sum_{k=1}^{m-1}\xi_k\Big)\prod_{l=1}^{m-1}\widehat{f_l}(\xi_l)\widehat{f_m}\Big(\xi-\sum_{k=1}^{m-1}\xi_k\Big)\,d\xi_1\cdots d\xi_{m-1}\,.$$

(b) Show that for functions $f_j\in\mathscr{S}(\mathbf{R}^n)$ we have

$$\int_{\mathbf{R}^n} T_\sigma(f_1,\dots,f_m)f_0\,dx=\int_{(\mathbf{R}^n)^m}\sigma(\vec{\xi}\,)\widehat{f_0}(-(\xi_1+\cdots+\xi_m))\widehat{f_1}(\xi_1)\cdots\widehat{f_m}(\xi_m)\,d\vec{\xi}\,.$$

7.5.4. Let $a>1$. Construct nonnegative smooth functions ϕ_j^m on $[0,\infty)^{m-1}$ that are supported in $[0,a^{m-1}]^{m-1}$ such that for all $(\xi_1,\dots,\xi_m)\in(\mathbf{R}^n)^m\setminus\{0\}$ we have

$$1=\sum_{j=1}^m \phi_j^m\Big(\frac{|\xi_1|}{|\xi_j|},\dots,\widehat{\frac{|\xi_j|}{|\xi_j|}},\dots,\frac{|\xi_m|}{|\xi_j|}\Big),$$

with the understanding that the variable with the hat is missing, and that $|\xi_i|/|\xi_j|=\infty$, when $\xi_j=0$ regardless of the value of ξ_i. Show that these functions provide a smooth decomposition of $(\mathbf{R}^n)^m\setminus\{0\}$ that is homogeneous of degree zero.
$\big[$*Hint:* Use induction. When $m=2$, pick a $\mathscr{C}_0^\infty$ function ϕ_1^2 with values in $[0,1]$ that is equal to 1 on $[0,1/a]$ and is supported in $[0,a]$, and define $\phi_2^2(t)=1-\phi_1^2(1/t)$. Assume that

$$1=\sum_{j=1}^{m-1}\phi_j^{m-1}\Big(\frac{|\xi_1|}{|\xi_j|},\dots,\widehat{\frac{|\xi_j|}{|\xi_j|}},\dots,\frac{|\xi_{m-1}|}{|\xi_j|}\Big)$$

for some smooth functions ϕ_j^{m-1} supported in $[0,a^{m-2}]^{m-2}$, $j=1,\dots,m-1$. Define

$$\phi_1^m\Big(\frac{|\xi_2|}{|\xi_1|},\dots,\frac{|\xi_m|}{|\xi_1|}\Big)=\prod_{k=2}^m\phi_1^2\Big(\frac{|\xi_k|}{|\xi_1|}\Big)$$

and for $j=2,\dots,m$ define

$$\phi_j^m\Big(\frac{|\xi_1|}{|\xi_j|},\dots,\widehat{\frac{|\xi_j|}{|\xi_j|}},\dots,\frac{|\xi_m|}{|\xi_j|}\Big)=\Big(1-\prod_{k=2}^m\phi_1^2\Big(\frac{|\xi_k|}{|\xi_1|}\Big)\Big)\phi_{j-1}^{m-1}\Big(\frac{|\xi_2|}{|\xi_j|},\dots,\widehat{\frac{|\xi_j|}{|\xi_j|}},\dots,\frac{|\xi_m|}{|\xi_j|}\Big),$$

which are supported in $[0,a^{m-1}]^{m-1}$. For instance, when $m=3$ these functions are

$$\begin{aligned}
\phi_1^3\Big(\frac{|\xi_2|}{|\xi_1|},\frac{|\xi_3|}{|\xi_1|}\Big)&=\phi_1^2\Big(\frac{|\xi_2|}{|\xi_1|}\Big)\phi_1^2\Big(\frac{|\xi_3|}{|\xi_1|}\Big)\\
\phi_2^3\Big(\frac{|\xi_1|}{|\xi_2|},\frac{|\xi_3|}{|\xi_2|}\Big)&=\Big(1-\phi_1^2\Big(\frac{1}{|\xi_1|/|\xi_2|}\Big)\phi_1^2\Big(\frac{|\xi_3|/|\xi_2|}{|\xi_1|/|\xi_2|}\Big)\Big)\phi_1^2\Big(\frac{|\xi_3|}{|\xi_2|}\Big)\\
\phi_3^3\Big(\frac{|\xi_1|}{|\xi_3|},\frac{|\xi_2|}{|\xi_3|}\Big)&=\Big(1-\phi_1^2\Big(\frac{|\xi_2|/|\xi_3|}{|\xi_1|/|\xi_3|}\Big)\phi_1^2\Big(\frac{1}{|\xi_1|/|\xi_3|}\Big)\Big)\phi_2^2\Big(\frac{|\xi_2|}{|\xi_3|}\Big).\big]
\end{aligned}$$