where A_k^t is the transpose of A_k. But $(A_k^t)^{-1} = A_k^t$ and $|A_k^t \vec{y}| \approx |\vec{y}|$; thus we have $w_\gamma(A_k^t \vec{y}) \approx w_\gamma(\vec{y})$. Therefore, by another change of variables, condition (7.5.37) is equivalent to

$$\sup_{j\in\mathbf{Z}} \int_{(\mathbf{R}^n)^m} \big|\big[\sigma(2^j\vec{\xi}\,)\widehat{\Psi_k}(\xi\,)\big]\widehat{\ }(\vec{y})\big|^2 w_\gamma(\vec{y})\,d\vec{y} < \infty. \tag{7.5.38}$$

Thus, condition (7.5.26) for σ^{*k} holds.

We now have that (7.5.26) holds for σ^{*k} for all Ψ in $\mathscr{S}_*((\mathbf{R}^n)^m)$. Theorem 7.5.5 implies that $(T_\sigma)^{*k}$, the kth adjoint of T_σ, is bounded from $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ whenever $2 < p_j < \infty$, in particular when $2 \le m < p_j < \infty$. In this case, each $(T_\sigma)^{*k}$ is bounded from $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$, with $1 < p < \infty$. By duality we obtain that T_σ is bounded from $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$, where $m < p_j < \infty$ when $j \ne k$ and $1 < p_k < m/(m-1)$. This is also valid when $m = 1$.

We now have boundedness for T_σ from $L^{q_1}(\mathbf{R}^n)\times\cdots\times L^{q_m}(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ in the following $m+1$ cases: (a) when all indices q_j are near infinity and (b) when the $m-1$ indices q_j, $j \ne k$ are near infinity and q_k is near 1 for all $k \in \{1,\dots,m\}$. Applying Corollary 7.2.4 we obtain that T_σ is bounded from $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for indices p_j satisfying $1 < p_1,\dots,p_m,p < \infty$. □

7.5.4 Proof of Main Result

In this section we discuss the proof of Theorem 7.5.5.

Proof. For each $j = 1,\dots,m$ we let R_j be the set of points $(\xi_1,\dots,\xi_m)$ in $(\mathbf{R}^n)^m$ such that $|\xi_j| = \max\{|\xi_1|,\dots,|\xi_m|\}$. For $j = 1,\dots,m$ we introduce nonnegative smooth functions ϕ_j on $[0,\infty)^{m-1}$ that are supported in $[0,\frac{11}{10}]^{m-1}$ such that

$$1 = \sum_{j=1}^m \phi_j\Big(\frac{|\xi_1|}{|\xi_j|},\dots,\widehat{\frac{|\xi_j|}{|\xi_j|}},\dots,\frac{|\xi_m|}{|\xi_j|}\Big)$$

for all $(\xi_1,\dots,\xi_m) \ne 0$, with the understanding that the variable with the hat is missing. These functions introduce a partition of unity of $(\mathbf{R}^n)^m \setminus \{0\}$ subordinate to a conical neighborhood of the region R_j. See Exercise 7.5.4.

Each region R_j can be written as the union of sets

$$R_{j,k} = \big\{(\xi_1,\dots,\xi_m) \in R_j : \ |\xi_k| \ge |\xi_s| \quad \text{for all } s \ne j\big\}$$

over $k = 1,\dots,m$. We need to work with a finer partition of unity, subordinate to a conical region of each $R_{j,k}$. To achieve this, for each j we introduce smooth functions $\phi_{j,k}$ on $[0,\infty)^{m-2}$ supported in $[0,\frac{11}{10}]^{m-2}$ such that