since the intersection of the annuli $\frac{6}{7}2^{k+b_1} \le |\xi| \le 2^{k+b_2+1}$ and $2^{b_1+l} \le |\xi| \le 2^{b_2+l}$ has measure zero if $l = r \bmod q$, $k = r \bmod q$, and l is not equal to k modulo q. The function G_r lies in L^2 by the assumption $\sum_k \|F_k\|_{L^2}^2 < \infty$, and thus part (a) yields that

$$\big\|G_r\big\|_{L^p} \le c(n,p,b_1,b_2,\Psi)\Big\|\Big(\sum_{k=r \mod q} |\Delta_k^\Omega(G_r)|^2\Big)^{1/2}\Big\|_{L^p}.$$

This inequality, combined with (7.5.13), implies (7.5.10), with G_r in place of $\sum_{k\in\mathbf{Z}} F_k$. Summing over $r \in \{0,1,\dots,q-1\}$ yields (7.5.10) with a bigger constant. □

7.5.2 Coifman-Meyer Method

In this subsection we describe a method to obtain boundedness for a bilinear multiplier operator using Fourier series expansions.

Theorem 7.5.3. *Suppose that a bounded function σ on $(\mathbf{R}^n)^2 \setminus \{(0,0)\}$ satisfies*

$$\big|\partial^{\alpha_1}\partial^{\alpha_2}\sigma(\xi_1,\xi_2)\big| \le C_{\alpha_1,\alpha_2}(|\xi_1|+|\xi_2|)^{-(|\alpha_1|+|\alpha_2|)} \tag{7.5.14}$$

for all $(\xi_1,\xi_2) \ne (0,0)$ and all multi-indices α_1,α_2, with $|\alpha_1|+|\alpha_2| \le 2n$. Given p_1,p_2,p such that $1 < p_1,p_2 \le \infty$ and $1/2 < p < \infty$ satisfying $1/p = 1/p_1 + 1/p_2$, the bilinear operator T_σ is bounded from $L^{p_1}(\mathbf{R}^n)\times L^{p_2}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$.

Proof. We first assume that $p_1,p_2 < \infty$. We fix a Schwartz function Ψ whose Fourier transform is nonnegative, supported in the set $\{\xi \in \mathbf{R}^n: \frac{6}{7} \le |\xi| \le 2\}$, is equal to 1 on the set $\{\xi \in \mathbf{R}^n: 1 \le |\xi| \le \frac{12}{7}\}$, and satisfies

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(\xi/2^j) = 1 \tag{7.5.15}$$

for all $\xi \ne 0$. We set $\widehat{\Phi}(\xi) = \sum_{j\le 0}\widehat{\Psi}(2^{-j}\xi)$ and define $\widehat{\Phi}(0) = 1$. Then $\widehat{\Phi}(\xi)$ is a smooth bump with compact support that is equal to 1 when $|\xi| \le \frac{12}{7}$ and is equal to zero when $|\xi| \ge 2$.

We introduce the Littlewood–Paley operators Δ_j^Ψ associated with Ψ via $\Delta_j^\Psi(f) = f*\Psi_{2^{-j}}$, and we fix Schwartz functions f_1,f_2 on $\mathbf{R}^n$. We express each f_j as

$$f_j = \sum_{k\in\mathbf{Z}} \Delta_k^\Psi(f_j)$$

where the sum is rapidly converging. We write

$$T_\sigma(f_1,f_2) = \sum_{k\in\mathbf{Z}}\Big[T_{\sigma_k^1}(f_1,f_2) + T_{\sigma_k^2}(f_1,f_2) + T_{\sigma_k^3}(f_1,f_2)\Big], \tag{7.5.16}$$