Then for all finitely simple functions f_k on X_k we have

$$\big\|T(f_1,\dots,f_m)\big\|_{L^q} \le M_0^{1-\theta}M_1^{\theta}\big\|f_1\big\|_{L^{p_1}}\cdots\big\|f_m\big\|_{L^{p_m}}. \tag{7.2.62}$$

Moreover, when $p_1,\dots,p_m<\infty$, the operator T_θ admits a unique bounded extension from $L^{p_1}(X_1,\mu_1)\times\cdots\times L^{p_m}(X_m,\mu_m)$ to $L^q(Y,\nu)$.

Proof. Take $T_z = T$ in Theorem 7.2.9, and use Exercise 1.3.8 in [156]. □

7.2.5 Multilinear Interpolation between Adjoint Operators

In this subsection we discuss a result that allows one to interpolate from a single estimate known for an operator and its adjoints. This theorem is useful in the setting where there is no duality, such as when an operator maps into L^q for $q<1$. For a number $q\in(0,\infty)$ set $q' = q/(q-1)$ when $q\neq 1$ and $1'=\infty$.

Theorem 7.2.12. *Let $0<p<\infty$, $A,B>0$, and let f be a measurable function on a σ-finite measure space (X,μ).*
(i) Suppose that $\|f\|_{L^{p,\infty}}\le A$. Then for every measurable set E of finite measure there exists a measurable subset E' of E with $\mu(E')\ge\mu(E)/2$ such that f is bounded on E' and

$$\left|\int_{E'} f\,d\mu\right| \le 2^{\frac{1}{p}}A\,\mu(E)^{1-\frac{1}{p}}. \tag{7.2.63}$$

(ii) Suppose that a measurable function f on X has the property that for any measurable subset E of X, with $\mu(E)<\infty$, there is a measurable subset E' of E, with $\mu(E')\ge\mu(E)/2$, such that f is integrable on E' and

$$\left|\int_{E'} f\,d\mu\right| \le B\,\mu(E)^{1-\frac{1}{p}}.$$

Then we have that

$$\|f\|_{L^{p,\infty}} \le B\,2^{\frac{2}{p}+\frac{3}{2}}. \tag{7.2.64}$$

Proof. Define $E' = E\setminus\{|f|>A\,2^{\frac{1}{p}}\mu(E)^{-\frac{1}{p}}\}$. Since the set $\{|f|>A\,2^{\frac{1}{p}}\mu(E)^{-\frac{1}{p}}\}$ has measure at most $\mu(E)/2$, it follows that $\mu(E')\ge\mu(E)/2$. Obviously, (7.2.63) holds for this choice of E'. This proves (i).

To prove (ii), write $X=\bigcup_{n=1}^\infty X_n$, with $\mu(X_n)<\infty$. Given $\alpha>0$, note that the measurable set $\{|f|>\alpha\}$ is contained in

$$\big\{\mathrm{Re}\, f>\tfrac{\alpha}{\sqrt2}\big\}\cup\big\{\mathrm{Im}\, f>\tfrac{\alpha}{\sqrt2}\big\}\cup\big\{\mathrm{Re}\, f<-\tfrac{\alpha}{\sqrt2}\big\}\cup\big\{\mathrm{Im}\, f<-\tfrac{\alpha}{\sqrt2}\big\}. \tag{7.2.65}$$

Let E_n be any of the preceding four sets intersected with X_n. By hypothesis, there is a measurable subset E_n' of E_n with measure at least $\mu(E_n)/2$. Then

$$\tfrac{\alpha}{2\sqrt2}\,\mu(E_n) \le \left|\int_{E_n'} f\,d\mu\right| \le B\,\mu(E_n)^{1-\frac{1}{p}}$$