implies that $f \in \mathscr{C}^{|\alpha|}$ for all $|\alpha| < \gamma$ and $\partial^\alpha f$ are bounded functions. It is straightforward to check the identity $\Delta_j^\Psi(\partial^\alpha f) = 2^{j|\alpha|}\Delta_j^{\partial^\alpha \Psi}(f)$. Notice that

$$2^{j(\gamma-|\alpha|)}\Delta_j^\Psi(\partial^\alpha f) = 2^{j\gamma}\Delta_j^{\partial^\alpha\Psi}(\Delta_{j-1}^\Psi + \Delta_j^\Psi + \Delta_{j+1}^\Psi)(f), \tag{1.4.32}$$

and from this it easily follows that

$$\sup_{j\geq 1} 2^{j(\gamma-|\alpha|)}\big\|\Delta_j^\Psi(\partial^\alpha f)\big\|_{L^\infty} \leq (2^\gamma+2)\,\big\|\partial^\alpha\Psi\big\|_{L^1}\sup_{j\geq 1}2^{j\gamma}\big\|\Delta_j^\Psi(f)\big\|_{L^\infty} < \infty. \tag{1.4.33}$$

Additionally, note that $S_0^\Phi(\partial^\alpha f) = \Phi * (\partial^\alpha f) = \partial^\alpha\Phi * f = \partial^\alpha\Phi * (\Phi + \Psi_{2^{-1}}) * f$, since the function $\widehat{\Phi} + \widehat{\Psi_{2^{-1}}}$ is equal to 1 on the support of $\widehat{\partial^\alpha\Phi}$. Thus

$$\begin{aligned}\big\|S_0^\Phi(\partial^\alpha f)\big\|_{L^\infty} &\leq \big\|\partial^\alpha\Phi\big\|_{L^1}\big(\big\|\Phi * f\big\|_{L^\infty} + \big\|\Psi_{2^{-1}} * f\big\|_{L^\infty}\big)\\ &\leq \big\|\partial^\alpha\Phi\big\|_{L^1}\Big(\Big\|S_0^\Phi(f)\Big\|_{L^\infty} + \sup_{j\geq 1}\big\|\Delta_j^\Psi(f)\big\|_{L^\infty}\Big),\end{aligned}$$

which, combined with (1.4.33), yields (1.4.31) for all $|\alpha| < \gamma$. □

Corollary 1.4.12. *Let $\gamma > 0$ and $f \in \Lambda_\gamma(\mathbf{R}^n)$. Then*

$$\big\|f\big\|_{\Lambda_\gamma} \approx \big\|f\big\|_{L^\infty} + \sum_{|\alpha| = [[\gamma]]}\big\|\partial^\alpha f\big\|_{\Lambda_{\gamma-[[\gamma]]}}. \tag{1.4.34}$$

Proof. One direction is a consequence of Corollary 1.4.11. For the other direction ($\lesssim$), we smoothly decompose $\widehat{\Psi} = \sum_{k=1}^n \widehat{\Psi^{(k)}}$ in (1.4.25) such that for ξ in the support of a given $\widehat{\Psi^{(k)}}$ we have $\xi_k \neq 0$. Introducing $\widehat{\zeta_k}(\xi) = \widehat{\Psi^{(k)}}(\xi)\xi_k^{-[[\gamma]]}$ write

$$\begin{aligned}\sup_{j\geq 1}2^{j\gamma}\big\|\Delta_j^\Psi(f)\big\|_{L^\infty} &\leq C\sum_{k=1}^n\sup_{j\geq 1}2^{j\gamma}2^{-j[[\gamma]]}\big\|\Delta_j^{\zeta_k}(\partial_k^{[[\gamma]]}f)\big\|_{L^\infty}\\ &\leq C\sum_{|\alpha|=[[\gamma]]}\sup_{j\geq 1}2^{j(\gamma-[[\gamma]])}\big\|\Delta_j^{\Psi_*}\Delta_j^{\zeta_k}(\partial^\alpha f)\big\|_{L^\infty},\end{aligned} \tag{1.4.35}$$

where we set $\widehat{\Psi_*}(\xi) = \widehat{\Psi}(2\xi) + \widehat{\Psi}(\xi) + \widehat{\Psi}(\frac{\xi}{2})$, which is equal to 1 on the support of $\widehat{\zeta_k}(\xi)$. As convolution with ζ_k is a bounded operator on L^∞, using (1.4.23), we obtain that (1.4.35) is at most the expression on the right in (1.4.34). □

We end this section by noting that the specific choice of the functions Ψ and Φ is unimportant in the Littlewood–Paley characterization of the spaces Λ_γ. In particular, if we know (1.4.25) and (1.4.8) for some choice of Littlewood–Paley operators $\Delta_j^{\widetilde{\Psi}}$ and some Schwartz function $\widetilde{\Phi}$ whose Fourier transform is supported in a neighborhood of the origin, then (1.4.25) and (1.4.8) also hold for our fixed Δ_j^Ψ and Φ.

Exercises

1.4.1. Fix $k \in \mathbf{Z}^+$, $\alpha_1, \dots, \alpha_n \in \mathbf{Z}^+ \cup \{0\}$, and $\gamma > 0$. Set $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
(a) Let $Q(x) = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ be a monomial on $\mathbf{R}^n$ of degree $|\alpha|$. Define $C(k,m) = \sum_{j=0}^k \binom{k}{j}(-1)^{k-j}j^m$ for $m \in \mathbf{Z}^+$. Show that when $|\alpha| \geq k$ for all $x, h \in \mathbf{R}^n$ we have

$$D_h^k(Q)(x) = \sum_{\substack{\beta\leq\alpha\\ |\beta|\geq k}}\binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n}C(k,|\beta|)\,h^\beta x^{\alpha-\beta}.$$