$$
\begin{aligned}
&\le \frac{1}{|I|}\big\|\mathscr{C}(f_0)\big\|_{L^r}\big\|\chi_I\big\|_{L^{r'}}\\
&\le \frac{\|\mathscr{C}\|_{L^r\to L^r}}{|I|}\big\|f_0\big\|_{L^r}|I|^{\frac{1}{r'}}\\
&\le C_r M_r(f)(x)\,,
\end{aligned}
$$

where we used the boundedness of the Carleson operator $\mathscr{C}$ from L^r to L^r.

We turn to the corresponding estimate for B_2. We have

$$
\begin{aligned}
B_2 &\le \frac{1}{|I|}\int_I\int_{\mathbf{R}}|f_\infty(z)|\left|\frac{1}{y-z}-\frac{1}{c_I-z}\right|dz\,dy\\
&= \frac{1}{|I|}\int_I\int_{(3I)^c}|f(z)|\left|\frac{y-c_I}{(y-z)(c_I-z)}\right|dz\,dy\\
&\le \int_I\left(\int_{(3I)^c}|f(z)|\,\frac{C}{(|c_I-z|+|I|)^2}\,dz\right)dy\\
&\le \int_I\frac{C}{|I|}M(f)(x)\,dy\\
&\le CM(f)(x)\\
&\le CM_r(f)(x)\,.
\end{aligned}
$$

This completes the proof of estimate (6.3.8). We now focus attention to the proof of (6.3.9). We derive estimate (6.3.9) as a consequence of Theorem 3.4.5, provided we have that

$$
\big\|M_d(\mathscr{C}(f))\big\|_{L^r(w)} \le \big\|M(\mathscr{C}(f))\big\|_{L^r(w)} < \infty\,. \tag{6.3.10}
$$

Unfortunately, the finiteness estimate (6.3.10) for general functions f in $L^p(w)$ cannot be easily deduced without a priori knowledge of the sought estimate (6.3.4) for $p=r$. However, we can show the validity of (6.3.10) for functions f with compact support and weights $w\in A_p$ that are bounded. This argument requires a few technicalities, which we now present. For a fixed constant B we introduce a truncated Carleson operator

$$
\mathscr{C}^B(f)=\sup_{|\xi|\le B}|H(M^\xi(f))|\,.
$$

Next we work with a weight w in A_p that is bounded. In fact, we work with $w_k=\min(w,k)$, which satisfies

$$
[w_k]_{A_p}\le c_p[w]_{A_p}
$$

for all $k\ge 1$ (see Exercise 7.1.8 in [156]). Finally, we take $f=h$ to be a smooth function with support contained in an interval $[-R,R]$. Then for $|\xi|\le B$ we have

$$
|H(M^\xi(h))(x)|\le 2R\big\|(M^\xi(h))'\big\|_{L^\infty}\chi_{|x|\le 2R}+\frac{\|h\|_{L^1}}{|x|+R}\chi_{|x|>2R}\le\frac{BC_hR}{|x|+R}\,,
$$