We apply (6.2.7) to the pair (E^j_α, F) for any $j = 1,2,3,4$. We find a subset $(E^j_\alpha)'$ of E^j_α of at least half its measure so that (6.2.7) holds for this pair. Then we have

$$\begin{aligned} \frac{\alpha}{2}|E^j_\alpha| \le \alpha|(E^j_\alpha)'| &\le \left| \int_{(E^j_\alpha)'} \Pi_N(\chi_F)(x)\,dx \right| \\ &\le 2C' \min(|E^j_\alpha|, |F|)\left(1 + \left|\log \frac{|E^j_\alpha|}{|F|}\right|\right). \end{aligned} \tag{6.2.8}$$

If $|E^j_\alpha| \le |F|$, this estimate implies that

$$|E^j_\alpha| \le |F|\,e\,e^{-\frac{1}{4C'}\alpha}, \tag{6.2.9}$$

while if $|E^j_\alpha| > |F|$, it implies that

$$\alpha \le 4C' \frac{|F|}{|E^j_\alpha|}\left(1 + \log \frac{|E^j_\alpha|}{|F|}\right). \tag{6.2.10}$$

Case 1: $\alpha > 4C'$. If $|E^j_\alpha| > |F|$, setting $t = |E^j_\alpha|/|F| > 1$ and using the fact that $\sup_{1<t<\infty} \frac{1}{t}(1+\log t) = 1$, we obtain that (6.2.10) fails. In this case we must therefore have that $|E^j_\alpha| \le |F|$. Applying (6.2.9) four times, we deduce

$$|\{|\Pi_N(\chi_F)| > 4\alpha\}| \le 4\,e\,|F|\,e^{-\frac{1}{4C'}\alpha}. \tag{6.2.11}$$

Case 2: $\alpha \le 4C'$. If $|E^j_\alpha| > |F|$, we use the elementary fact that if $t > 1$ satisfies $t(1+\log t)^{-1} < \frac{B}{\alpha}$, then $t < \frac{2B}{\alpha}(1 + \log \frac{2B}{\alpha})$; to prove this fact one may use the inequalities $t < \frac{2B}{\alpha}(1+\log\sqrt{t})$ and $\log\sqrt{t} \le \log t - \log(1+\log\sqrt{t}) \le \log \frac{2B}{\alpha}$ for $t > 1$. Taking t = $|E^j_\alpha|/|F|$ and $B = 4C'$ in (6.2.10) yields

$$\frac{|E^j_\alpha|}{|F|} \le \frac{8C'}{\alpha}\left(1 + \log \frac{8C'}{\alpha}\right). \tag{6.2.12}$$

If $|E^j_\alpha| \le |F|$, then we use (6.2.9), but we note that for some constant $c' > 1$ we have

$$e\,e^{-\frac{1}{4C'}\alpha} \le c'\frac{8C'}{\alpha}\left(1 + \log \frac{8C'}{\alpha}\right)$$

whenever $\alpha \le 4C'$. Thus, when $\alpha \le 4C'$, we always have

$$|\{|\Pi_N(\chi_F)| > 4\alpha\}| \le c'\frac{32C'}{\alpha}|F|\left(1 + \log \frac{8C'}{\alpha}\right). \tag{6.2.13}$$

Combining (6.2.11) and (6.2.13), we obtain (6.2.1) with Π_N in place of $\mathscr{C}$. Then (6.2.6) yields (6.2.1) with $\mathscr{C}_N$ in place of $\mathscr{C}$ and this suffices for the proof of the statement in part (a) of the theorem by a limiting argument, as observed before.