and thus

$$\begin{aligned}|\langle\partial^\alpha h-u_\alpha,\varphi\rangle| &\le |\langle\partial^\alpha h-\partial^\alpha h_k,\varphi\rangle|+|\langle\partial^\alpha h_k-u_\alpha,\varphi\rangle|\\ &=|\langle h-h_k,\partial^\alpha\varphi\rangle|+|\langle\partial^\alpha h_k-u_\alpha,\varphi\rangle|<\varepsilon.\end{aligned}$$

Since ε was arbitrary, we deduce that $\partial^\alpha h=u_\alpha$, in particular $h\in\mathscr{C}^N$. □

Corollary 1.4.8. *Any function f in $\dot\Lambda_\gamma$ lies in $\mathscr{C}^{|\beta|}$ for any $|\beta|<\gamma$, and its derivatives $\partial^\beta f$ lie in $\dot\Lambda_{\gamma-|\beta|}$ and satisfy*

$$\big\|\partial^\beta f\big\|_{\dot\Lambda_{\gamma-|\beta|}}\le C_{n,\gamma,\beta}\big\|f\big\|_{\dot\Lambda_\gamma}. \tag{1.4.21}$$

Proof. We proved in Theorem 1.4.6 that if f lies in $\dot\Lambda_\gamma$, then (1.4.7) holds, and that (1.4.7) implies that there exists a polynomial Q such that $f-Q$ lies in $\mathscr{C}^{[[\gamma]]}$ and in $\dot\Lambda_\gamma$. It follows that f lies in $\mathscr{C}^{[[\gamma]]}$. It also follows that Q lies in $\dot\Lambda_\gamma$, and this imposes a restriction on the degree of Q; in view of the result of Exercise 1.4.1, we have that Q must have degree at most $[\gamma]$; thus, $f\equiv f-Q$ in the space $\mathscr{S}'/\mathscr{P}_{[\gamma]}$, i.e., they belong to the same equivalence class.

Let Ψ be a Schwartz function on $\mathbf{R}^n$ whose Fourier transform is supported in $1-\frac{1}{7}\le|\xi|\le 2$ and is equal to one on $1\le|\xi|\le 2-\frac{2}{7}$. Given a multi-index β with $|\beta|<\gamma$, we denote by $\Delta_j^{\partial^\beta\Psi}$ the Littlewood–Paley operator associated with $(\partial^\beta\Psi)_{2^{-j}}$. Then one has

$$\Delta_j^\Psi(\partial^\beta f)=2^{j|\beta|}\Delta_j^{\partial^\beta\Psi}(f)$$

for all $f\in\Lambda_\gamma$. One can easily check that

$$2^{j(\gamma-|\beta|)}\Delta_j^\Psi(\partial^\beta f)=2^{j\gamma}\Delta_j^{\partial^\beta\Psi}(\Delta_{j-1}^\Psi+\Delta_j^\Psi+\Delta_{j+1}^\Psi)(f),$$

and from this it easily follows that

$$\sup_{j\in\mathbf{Z}}2^{j(\gamma-|\beta|)}\big\|\Delta_j^\Psi(\partial^\beta f)\big\|_{L^\infty}\le(2^\gamma+1+2^{-\gamma})\,\big\|\partial^\beta\Psi\big\|_{L^1}\sup_{j\in\mathbf{Z}}2^{j\gamma}\big\|\Delta_j^\Psi(f)\big\|_{L^\infty},$$

which implies that $\partial^\beta f$ lies in $\dot\Lambda_{\gamma-|\beta|}$ when $|\beta|<\gamma$. □

1.4.3 Littlewood–Paley Characterization of Inhomogeneous Lipschitz Spaces

We have seen that quantities involving the Littlewood–Paley operators Δ_j characterize homogeneous Lipschitz spaces. We now address the same question for inhomogeneous Lipschitz spaces.

We fix a radial Schwartz function Ψ whose Fourier transform $\widehat{\Psi}$ is nonnegative, is supported in the annulus $1-\frac{1}{7} \le |\xi| \le 2$, is equal to one on the annulus $1 \le |\xi| \le 2-\frac{2}{7}$, and satisfies

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$$

for all $\xi \neq 0$. We define a Schwartz function Φ introduced by setting

$$\widehat{\Phi}(\xi) = \begin{cases} \sum_{j\le 0} \widehat{\Psi}(2^{-j}\xi) & \text{when } \xi \neq 0, \\ 1 & \text{when } \xi = 0. \end{cases} \tag{1.4.22}$$

Note that $\widehat{\Phi}(\xi)$ is equal to 1 for $|\xi| \le 2-\frac{2}{7}$ and vanishes when $|\xi| \ge 2$. Finally, we define $\Delta_j^{\Psi}(f) = \Psi_{2^{-j}} * f$ and $S_0^{\Phi}(f) = \Phi * f$ for any $f \in \mathscr{S}'(\mathbf{R}^n)$.

Theorem 1.4.9. *Let Ψ, Φ, Δ_j^{Ψ}, and S_0^{Φ} be as above, and let $\gamma > 0$. Then there is a constant $C = C(n,\gamma)$ such that for every function f in Λ_γ the following estimate holds:*

$$\big\|S_0^{\Phi}(f)\big\|_{L^\infty} + \sup_{j\ge 1} 2^{j\gamma}\big\|\Delta_j^{\Psi}(f)\big\|_{L^\infty} \le C\big\|f\big\|_{\Lambda_\gamma}. \tag{1.4.23}$$

Conversely, suppose that a tempered distribution f satisfies

$$\big\|S_0^{\Phi}(f)\big\|_{L^\infty} + \sup_{j\ge 1} 2^{j\gamma}\big\|\Delta_j^{\Psi}(f)\big\|_{L^\infty} < \infty. \tag{1.4.24}$$

Then f is in $\mathscr{C}^{[[\gamma]]}$, and the derivatives $\partial^\alpha f$ are bounded for all $|\alpha| \le [[\gamma]]$. Moreover, f lies in Λ_γ, and there is a constant $C' = C'(n,\gamma)$ such that

$$\big\|f\big\|_{\Lambda_\gamma} \le C'\Big(\big\|S_0^{\Phi}(f)\big\|_{L^\infty} + \sup_{j\ge 1} 2^{j\gamma}\big\|\Delta_j^{\Psi}(f)\big\|_{L^\infty}\Big). \tag{1.4.25}$$

In particular, functions in Λ_γ are in $\mathscr{C}^{[[\gamma]]}$ and have bounded derivatives up to order $[[\gamma]]$. ~~*Also,*~~

$$\big\|f\big\|_{\Lambda_\gamma} \approx \sum_{|\alpha|<[\gamma]} \big\|\partial^\alpha f\big\|_{L^\infty} + \sum_{|\alpha|=[\gamma]} \big\|\partial^\alpha f\big\|_{\Lambda_{\gamma-[\gamma]}}.$$

Proof. The proof of (1.4.23) is immediate since we trivially have

$$\big\|S_0^{\Phi}(f)\big\|_{L^\infty} = \big\|f * \Phi\big\|_{L^\infty} \le \big\|\Phi\big\|_{L^1}\big\|f\big\|_{L^\infty} \le C\big\|f\big\|_{\Lambda_\gamma},$$

and, in view of estimate (1.4.11), we have

$$\sup_{j\ge 1} 2^{j\gamma}\big\|\Delta_j^{\Psi}(f)\big\|_{L^\infty} \le C\big\|f\big\|_{\dot{\Lambda}_\gamma} \le C\big\|f\big\|_{\Lambda_\gamma}.$$

We may therefore focus on the proof of the converse estimate (1.4.25). We fix $f \in \mathscr{S}'(\mathbf{R}^n)$ which satisfies (1.4.24). We introduce Schwartz functions ζ, η such that

$$\widehat{\zeta}(\xi)^2 + \sum_{j=1}^{\infty} \widehat{\eta}(2^{-j}\xi)^2 = 1$$

and such that $\widehat{\eta}$ is supported in the annulus $\frac{2}{5} \le |\xi| \le 1$ and $\widehat{\zeta}$ is supported in the ball $|\xi| \le 1$. We associate Littlewood–Paley operators Δ_j^{η} given by convolution with the functions $\eta_{2^{-j}}$ and we let $\Delta_j^{\Theta} = \Delta_{j-1}^{\Psi} + \Delta_j^{\Psi} + \Delta_{j+1}^{\Psi}$. Using this identity and (1.4.24) we obtain for some $C_0 < \infty$

$$\|\Delta_j^{\Theta}(f)\|_{L^\infty} \le C_0 2^{-j\gamma}. \tag{1.4.26}$$

Note that $\widehat{\Phi}$ is equal to one on the support of $\widehat{\zeta}$. Moreover, $\Delta_j^{\Theta}\Delta_j^{\eta} = \Delta_j^{\eta}$; hence, for our given tempered distribution f we have the identity

$$f = \zeta * \zeta * \Phi * f + \sum_{j=1}^{\infty} \eta_{2^{-j}} * \eta_{2^{-j}} * \Delta_j^{\Theta}(f), \tag{1.4.27}$$

where the series converges in $\mathscr{S}'(\mathbf{R}^n)$, in view of the result of Exercise 1.1.5.

But this series also converges in L^∞ since, in view of (1.4.26),

$$\big\|\eta_{2^{-j}} * \eta_{2^{-j}} * \Delta_j^{\Theta}(f)\|_{L^\infty} \le \|\eta * \eta\|_{L^1} \|\Delta_j^{\Theta}(f)\|_{L^\infty} \le C_0 2^{-j\gamma},$$

and thus f is a continuous and bounded function. Also, for all $|\alpha| < \gamma$ we have

$$\big\|\partial^{\alpha}(\eta_{2^{-j}} * \eta_{2^{-j}} * \Delta_j^{\Theta}(f))\|_{L^\infty} \le 2^{j|\alpha|} \|\partial^{\alpha}(\eta * \eta)\|_{L^1} \|\Delta_j^{\Theta}(f)\|_{L^\infty} \le C_0 2^{-j(\gamma-|\alpha|)},$$

and thus summing over j yields a finite constant. Proposition 1.1.5 applies and yields that our given tempered distribution f is a $\mathscr{C}^{|\alpha|}$ function whose derivatives are bounded for all $|\alpha| < \gamma$, or equivalently, for all $|\alpha| \le [[\gamma]]$.

It remains to show that the function f is in Λ_γ. With $k = [\gamma]$ we write

$$\frac{D_h^{k+1}(f)}{|h|^{\gamma}} = \zeta * \frac{D_h^{k+1}(\zeta)}{|h|^{\gamma}} * \Phi * f + \sum_{j=1}^{\infty} \eta_{2^{-j}} * \frac{D_h^{k+1}(\eta_{2^{-j}})}{|h|^{\gamma}} * \Delta_j^{\Theta}(f). \tag{1.4.28}$$

We use Proposition 1.4.5 to estimate the L^∞ norm of the term $\zeta * \frac{D_h^{k+1}(\zeta)}{|h|^{\gamma}} * \Phi * f$ in the previous sum as follows:

$$\begin{aligned} \Big\|\zeta * \tfrac{D_h^{k+1}(\zeta)}{|h|^{\gamma}} * \Phi * f\Big\|_{L^\infty} &\le \Big\|\tfrac{D_h^{k+1}(\zeta)}{|h|^{\gamma}}\Big\|_{L^\infty} \Big\|\zeta * \Phi * f\Big\|_{L^1} \\ &\le C' \min\big(\tfrac{1}{|h|^{\gamma}}, \tfrac{|h|^{k+1}}{|h|^{\gamma}}\big) \big\|\Phi * f\big\|_{L^\infty} \\ &\le C' \big\|\Phi * f\big\|_{L^\infty}. \end{aligned} \tag{1.4.29}$$