6.1.5. Prove that there is a constant $C>0$ such that for any interval J and any $b>0$,

$$\int_J\int_{J^c}\frac{1}{(1+\frac{|x-y|}{b|J|})^{20}}\,dx\,dy = Cb^2|J|^2\,.$$

[*Hint:* Translate J to the interval $[-\frac{1}{2}|J|,\frac{1}{2}|J|]$ and change variables.]

6.1.6. Let φ_s be as in (6.1.4). Let $\mathbf{T}_2$ be a 2-tree and $f\in L^2(\mathbf{R})$.
(a) Show that there is a constant C such that for all sequences of complex scalars $\{\lambda_s\}_{s\in\mathbf{T}_2}$ we have

$$\Big\|\sum_{s\in\mathbf{T}_2}\lambda_s\,\varphi_s\Big\|_{L^2(\mathbf{R})}\le C\Big(\sum_{s\in\mathbf{T}_2}|\lambda_s|^2\Big)^{\frac{1}{2}}.$$

(b) Use duality to conclude that

$$\sum_{s\in\mathbf{T}_2}\big|\langle f\,|\,\varphi_s\rangle\big|^2\le C^2\big\|f\big\|_{L^2}^2\,.$$

[*Hint:* To prove part (a) define $\mathscr{G}_m=\{s\in\mathbf{T}_2:\ |I_s|=2^m\}$. Then for $s\in\mathscr{G}_m$ and $s'\in\mathscr{G}_{m'}$, the functions φ_s and $\varphi_{s'}$ are orthogonal to each other, and it suffices to obtain the corresponding estimate when the summation is restricted to a given $\mathscr{G}_m$. But for s in $\mathscr{G}_m$, the intervals I_s are disjoint, and we may use the idea of the proof of Lemma 6.1.2. Use that $\sum_{u:\,\omega_u=\omega_s}\big|\langle\varphi_s\,|\,\varphi_u\rangle\big|\le C$ for every fixed s.]

6.1.7. Fix $A\ge 1$. Let $\mathbf{S}$ be a finite collection of dyadic tiles such that for all s_1, s_2 in $\mathbf{S}$ we have either $\omega_{s_1}\cap\omega_{s_2}=\emptyset$ or $AI_{s_1}\cap AI_{s_2}=\emptyset$. Let $N_{\mathbf{S}}$ be the *counting function* of $\mathbf{S}$, defined by

$$N_{\mathbf{S}}=\sup_{x\in\mathbf{R}}\#\{I_s:\ s\in\mathbf{S}\text{ and }x\in I_s\}.$$

(a) Show that for any $M>2$ there exists a $C_M>0$ such that for all $f\in L^2(\mathbf{R})$ we have

$$\sum_{s\in\mathbf{S}}\bigg|\Big\langle f,|I_s|^{-\frac{1}{2}}\Big(1+\frac{\operatorname{dist}(\cdot,I_s)}{|I_s|}\Big)^{-\frac{M}{2}}\Big\rangle\bigg|^2\le C_M N_{\mathbf{S}}\big\|f\big\|_{L^2}^2\,.$$

(b) Let φ_s be as in (6.1.4). Show that for any $M>2$ there exists a $C_M>0$ such that for all finite sequences of scalars $\{a_s\}_{s\in\mathbf{S}}$ we have

$$\Big\|\sum_{s\in\mathbf{S}}a_s\varphi_s\Big\|_{L^2}^2\le C_M(1+A^{-M}N_{\mathbf{S}})\sum_{s\in\mathbf{S}}|a_s|^2\,.$$

(c) Conclude that for any $M>2$ there exists a $C_M>0$ such that for all $f\in L^2(\mathbf{R})$ we have

$$\sum_{s\in\mathbf{S}}\big|\langle f,\varphi_s\rangle\big|^2\le C_M(1+A^{-M}N_{\mathbf{S}})\big\|f\big\|_{L^2}^2.$$

[*Hint:* Use the idea of Lemma 6.1.2 to prove part (a) when $N_{\mathbf{S}}=1$. So we may now suppose that $N_{\mathbf{S}}>1$. Call an element $s\in\mathbf{S}$ vertically-maximal if the region

$R_s = \{(x,y) \in \mathbf{R}^2 : \inf I_s < x < \sup I_s,\ y \geq \sup \omega_s\}$ does not intersect any other tile $s' \in \mathbf{S}$. Let $\mathbf{S}_1$ be the set of all vertically-maximal tiles in $\mathbf{S}$. Then $N_{\mathbf{S}_1} = 1$; otherwise, some $x \in \mathbf{R}$ would belong to both I_s and $I_{s'}$ for $s \neq s' \in \mathbf{S}_1$, and thus R_s and $R_{s'}$ would have to intersect, contradicting the vertical-maximality of $\mathbf{S}_1$. Now define $\mathbf{S}_2$ to be the set of all vertically-maximal tiles in $\mathbf{S} \setminus \mathbf{S}_1$. As before, we have $N_{\mathbf{S}_2} = 1$. Continue in this way and write $\mathbf{S}$ as a union of at most $N_{\mathbf{S}}$ families of tiles $\mathbf{S}_j$, each of which has the property $N_{\mathbf{S}_j} = 1$. Apply the result to each $\mathbf{S}_j$ and then sum over j. Part (b): observe that whenever $s_1, s_2 \in \mathbf{S}$ and $s_1 \neq s_2$ we must have either $\langle \varphi_{s_1}, \varphi_{s_2} \rangle = 0$ or $\operatorname{dist}(I_{s_1}, I_{s_2}) \geq (A-1)\max(|I_{s_1}|, |I_{s_2}|)$, which implies

$$\Big(1 + \frac{\operatorname{dist}(I_{s_1}, I_{s_2})}{\max(|I_{s_1}|, |I_{s_2}|)}\Big)^{-M} \leq A^{-\frac{M}{2}} \Big(1 + \frac{\operatorname{dist}(I_{s_1}, I_{s_2})}{\max(|I_{s_1}|, |I_{s_2}|)}\Big)^{-\frac{M}{2}}.$$

Use this estimate to obtain

$$\Big\| \sum_{s \in \mathbf{S}} a_s \varphi_s \Big\|_{L^2}^2 \leq \sum_{s \in \mathbf{S}} |a_s|^2 + \frac{C_M}{A^{\frac{M}{2}}} \bigg\| \sum_{s \in \mathbf{S}} \frac{|a_s|}{|I_s|^{\frac{1}{2}}} \Big(1 + \frac{\operatorname{dist}(x, I_s)}{|I_s|}\Big)^{-\frac{M}{2}} \bigg\|_{L^2}^2$$

by expanding the square on the left. The required estimate follows from the dual statement to part (a). Part (c) follows from part (b) by duality.]

6.1.8. Let φ_s be as in (6.1.4) and let $\mathbf{D}_m$ be the set of all dyadic tiles s with $|I_s| = 2^m$. Show that there is a constant C (independent of m) such that for square-integrable sequences of scalars $\{a_s\}_{s \in \mathbf{D}_m}$ we have

$$\Big\| \sum_{s \in \mathbf{D}_m} a_s \varphi_s \Big\|_{L^2}^2 \leq C \sum_{s \in \mathbf{D}_m} |a_s|^2 .$$

Conclude from this that

$$\sum_{s \in \mathbf{D}_m} \big|\langle f, \varphi_s \rangle\big|^2 \leq C \big\|f\big\|_{L^2}^2 .$$

6.1.9. Fix $c_0 > 0$ and a Schwartz function φ whose Fourier transform is supported in the interval $[-\frac{3}{8}, \frac{3}{8}]$ and that satisfies

$$\sum_{l \in \mathbf{Z}} |\widehat{\varphi}(t + \tfrac{l}{2})|^2 = c_0$$

for all real numbers t. Define functions φ_s as follows. Fix an integer m and set

$$\varphi_s(x) = 2^{-\frac{m}{2}} \varphi(2^{-m}x - k) e^{2\pi i 2^{-m} x \frac{l}{2}}$$

whenever $s = [k2^m, (k+1)2^m) \times [l2^{-m}, (l+1)2^{-m})$ is a tile in $\mathbf{D}_m$.
(a) Prove that for all Schwartz functions f we have

$$\sum_{s \in \mathbf{D}_m} \big|\langle f \,|\, \varphi_s \rangle\big| < \infty .$$