Since all s that appear in the definition of F_{2J} satisfy $|\omega_s| \le (4|J|)^{-1}$, it follows that we have the estimate

$$
\begin{aligned}
|F_{2J}(x)| &\le 2\chi_E(x) \sup_{\delta > |\omega_{u_x}|^{-1}} \int_{\mathbf{R}} \Big| \sum_{s\in\mathbf{T}_2} \varepsilon_s \langle g \,|\, \varphi_s\rangle \varphi_s(z)\Big| \tfrac{1}{\delta}\Big|\psi\big(\tfrac{x-z}{\delta}\big)\Big|\, dz \\
&\le C \sup_{\delta > 4|J|} \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} \Big|\sum_{s\in\mathbf{T}_2} \varepsilon_s \langle g \,|\, \varphi_s\rangle \varphi_s(z)\Big|\, dz. \qquad (6.1.59)
\end{aligned}
$$

(The last inequality follows from Exercise 2.1.14 in [156].) Observe that the maximal function in (6.1.59) satisfies the property

$$
\sup_{x\in J}\sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}|h(t)|\,dt \le 2\inf_{x\in J}\sup_{\delta>2|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}|h(t)|\,dt\,.
$$

Using this property, we obtain

$$
\begin{aligned}
\Sigma_{22} &\le \sum_{J\in\mathscr{J}} \big\|F_{2J}\big\|_{L^1(J)} \le \sum_{J\in\mathscr{J}} \big\|F_{2J}\big\|_{L^\infty(J)}|G_J| \\
&\le C \sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3I_{\mathrm{top}(\mathbf{T})}}} |E|\mathscr{M}(E;\mathbf{T})|J| \sup_{x\in J}\sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\Big|\sum_{s\in\mathbf{T}_2}\varepsilon_s\langle g\,|\,\varphi_s\rangle\varphi_s(z)\Big|\,dz \\
&\le 2C|E|\mathscr{M}(E;\mathbf{T}) \sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3I_{\mathrm{top}(\mathbf{T})}}} \int_J \sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\Big|\sum_{s\in\mathbf{T}_2}\varepsilon_s\langle g\,|\,\varphi_s\rangle\varphi_s(z)\Big|\,dz\,dx \\
&\le C|E|\mathscr{M}(E;\mathbf{T})\Big\|M\Big(\sum_{s\in\mathbf{T}_2}\varepsilon_s\langle g\,|\,\varphi_s\rangle\varphi_s\Big)\Big\|_{L^1(3I_{\mathrm{top}(\mathbf{T})})},
\end{aligned}
$$

where M is the Hardy–Littlewood maximal operator. Using the Cauchy–Schwarz inequality and the boundedness of M on $L^2(\mathbf{R})$, we obtain the following estimate:

$$
\Sigma_{22} \le C|E|\mathscr{M}(E;\mathbf{T})\,|I_{\mathrm{top}(\mathbf{T})}|^{\frac12}\Big\|\sum_{s\in\mathbf{T}_2}\varepsilon_s\langle g\,|\,\varphi_s\rangle\varphi_s\Big\|_{L^2}.
$$

Appealing to the result of Exercise 6.1.6(a), we deduce

$$
\Big\|\sum_{s\in\mathbf{T}_2}\varepsilon_s\langle g\,|\,\varphi_s\rangle\varphi_s\Big\|_{L^2} \le C\Big(\sum_{s\in\mathbf{T}_2}\big|\varepsilon_s\langle g\,|\,\varphi_s\rangle\big|^2\Big)^{\frac12} \le C'|I_{\mathrm{top}(\mathbf{T})}|^{\frac12}\,\mathscr{E}(g;\mathbf{T})\,.
$$

The first estimate was also shown in (6.1.46); the same argument applies here, and the presence of the ε_s's does not introduce any change. We conclude that

$$
\Sigma_{22} \le C|E|\mathscr{M}(E;\mathbf{T})|I_{\mathrm{top}(\mathbf{T})}|\,\mathscr{E}(g;\mathbf{T})\,,
$$

which is what we needed to prove. This completes the proof of Lemma 6.1.10. □

The proof of the theorem is now complete.