with the interpretation that $2^{-1}I_u = \emptyset$. ($2^k I_u$ has the same center as I_u but 2^k times its length.) It follows that for all u in $\mathbf{U}_{\max}$ there exists an integer $k \geq 0$ such that

$$|E|\,\frac{\mu}{8}\,|I_u|\,2^{-k} < \int_{E\cap N^{-1}[\omega_u]\cap\left(2^k I_u\setminus 2^{k-1}I_u\right)} \frac{dx}{(1+\frac{|x-c(I_u)|}{|I_u|})^{10}} \leq \frac{|E\cap N^{-1}[\omega_u]\cap 2^k I_u|}{(\frac{4}{5})^{10}(1+2^{k-2})^{10}}\,.$$

We therefore conclude that

$$\mathbf{U}_{\max} = \bigcup_{k=0}^{\infty} \mathbf{U}_k\,,$$

where

$$\mathbf{U}_k = \{u \in \mathbf{U}_{\max} :\ |I_u| \leq 8\cdot 5^{10}\,2^{-9k}\,\mu^{-1}\,|E|^{-1}|E\cap N^{-1}[\omega_u]\cap 2^k I_u|\}\,.$$

The required estimate (6.1.42) will be a consequence of the sequence of estimates

$$\sum_{u\in\mathbf{U}_k} |I_u| \leq C\,2^{-8k}\mu^{-1}\,, \qquad k \geq 0\,. \tag{6.1.43}$$

We now fix a $k \geq 0$ and we concentrate on (6.1.43). Select an element $v_0 \in \mathbf{U}_k$ such that $|I_{v_0}|$ is the largest possible among elements of $\mathbf{U}_k$. Then select an element $v_1 \in \mathbf{U}_k \setminus \{v_0\}$ such that the enlarged rectangle $(2^k I_{v_1}) \times \omega_{v_1}$ is disjoint from the enlarged rectangle $(2^k I_{v_0}) \times \omega_{v_0}$ and $|I_{v_1}|$ is the largest possible. Continue this process by induction. At the jth step select an element of

$$\mathbf{U}_k \setminus \{v_0, \dots, v_{j-1}\}$$

such that the enlarged rectangle $(2^k I_{v_j}) \times \omega_{v_j}$ is disjoint from all the enlarged rectangles of the previously selected tiles and the length $|I_{v_j}|$ is the largest possible. This process will terminate after a finite number of steps. We denote by $\mathbf{V}_k$ the set of all selected tiles in $\mathbf{U}_k$.

We make a few observations. Recall that all elements of $\mathbf{U}_k$ are maximal rectangles in $\mathbf{U}$ and therefore disjoint. For any $u \in \mathbf{U}_k$ there exists a selected $v \in \mathbf{V}_k$ with $|I_u| \leq |I_v|$ such that the enlarged rectangles corresponding to u and v intersect. Let us associate this u to the selected v. Observe that if u and u' are associated with the same selected v, they are disjoint, and since both ω_u and $\omega_{u'}$ contain ω_v, the intervals I_u and $I_{u'}$ must be disjoint. Thus, tiles $u \in \mathbf{U}_k$ associated with a fixed $v \in \mathbf{V}_k$ have disjoint I_u's and satisfy

$$I_u \subseteqq 2^{k+2} I_v\,.$$

Consequently,

$$\sum_{\substack{u\in\mathbf{U}_k\\ u \text{ associated with } v}} |I_u| \leq |2^{k+2} I_v| = 2^{k+2}|I_v|\,.$$