We now write $f_L = f_L^1 + f_L^2$, where

$$f_L^1 = \sum_{j=-L}^{-1} \Delta_j^\Theta(f) * \eta_{2^{-j}} * \eta_{2^{-j}}\,, \qquad\qquad f_L^2 = \sum_{j=0}^{L} \Delta_j^\Theta(f) * \eta_{2^{-j}} * \eta_{2^{-j}}\,.$$

It follows from (1.4.13) that with $C_0' = (2^\gamma + 1 + 2^{-\gamma})C_0$ we have

$$\|\Delta_j^\Theta(f) * \eta_{2^{-j}} * \eta_{2^{-j}}\|_{L^\infty} \le \|\Delta_j^\Theta(f)\|_{L^\infty}\|\eta_{2^{-j}} * \eta_{2^{-j}}\|_{L^1} \le C_0'\|\eta * \eta\|_{L^1} 2^{-j\gamma};$$

thus, f_L^2 converges uniformly to a continuous and bounded function g_2 as $L \to \infty$. Also, $\partial^\beta f_L^2$ converges uniformly for all $|\beta| < \gamma$ as $L \to \infty$. Using Lemma 1.4.7 we conclude that g_2 is in $\mathscr{C}^{[[\gamma]]}$ and that $\partial^\beta f_L^2$ converges uniformly to $\partial^\beta g_2$ as $L \to \infty$ for all $|\beta| < \gamma$.

We now turn our attention to f_L^1. Obviously, f_L^1 is in $\mathscr{C}^\infty$ and

$$\partial^\alpha f_L^1 = \sum_{j=-L}^{-1} \Delta_j^\Theta(f) * 2^{j|\alpha|}(\partial^\alpha \eta)_{2^{-j}} * \eta_{2^{-j}}\,.$$

Thus for all multi-indices α with $|\alpha| \ge [\gamma] + 1$ we have

$$\sup_{L\in\mathbf{Z}^+} \|\partial^\alpha f_L^1\|_{L^\infty} \le \sum_{j=-\infty}^{-1} C_0' 2^{-j\gamma} 2^{j([\gamma]+1)} \|\partial^\alpha \eta * \eta\|_{L^1} = c_{\alpha,\gamma} C_0 < \infty\,. \tag{1.4.19}$$

Let P_L^d be the Taylor polynomial of f_L^1 of degree d. By Taylor's theorem we have

$$f_L^1(x) - P_L^{[\gamma]}(x) = ([\gamma]+1) \sum_{|\alpha|=[\gamma]+1} \frac{x^\alpha}{\alpha!} \int_0^1 (1-t)^{[\gamma]} (\partial^\alpha f_L^1)(tx)\, dt\,. \tag{1.4.20}$$

Using (1.4.19), with $|\alpha| \in \{[\gamma]+1, \dots, [\gamma]+|\beta|+2\}$, we obtain that the sequence $\{\nabla(\partial^\beta(f_L^1 - P_L^{[\gamma]}))\}_{L=1}^\infty$ is uniformly bounded on every ball $\overline{B(0,K)}$; thus, the sequence $\{\partial^\beta(f_L^1 - P_L^{[\gamma]})\}_{L=1}^\infty$ is equicontinuous on every such ball. By the Arzelà–Ascoli theorem, for every $K = 1, 2, \dots$ and for every $|\beta| < \gamma$ there is a subsequence of $\{\partial^\beta(f_L^1 - P_L^{[\gamma]})\}_{L=1}^\infty$ that converges uniformly on $\overline{B(0,K)}$. The diagonal subsequence of these subsequences converges uniformly on every compact subset of $\mathbf{R}^n$ for all $|\beta| < \gamma$. Hence, there is a continuous function g_1 on $\mathbf{R}^n$ and a subsequence L_m of $\mathbf{Z}^+$ such that $f_{L_m}^1 - P_{L_m}^{[\gamma]} \to g_1$ uniformly on compact sets as $m \to \infty$ and $\partial^\beta(f_{L_m}^1 - P_{L_m}^{[\gamma]})$ converges uniformly on compact sets for all $|\beta| \ge [\gamma]$. Using Lemma 1.4.7, stated at the end of this proof, we deduce that g_1 is $\mathscr{C}^{[\gamma]}$ and that $\partial^\beta(f_{L_m}^1 - P_{L_m}^{[\gamma]}) \to \partial^\beta g_1$ as $m \to \infty$ for all $|\beta| \le [\gamma]$.

Set $g = g_1 + g_2$. It follows from (1.4.20) and from $\sup_{L\in\mathbf{Z}^+} \|f_L^2\|_{L^\infty} < \infty$ that $|g(x)| \le C_{n,\gamma} C_0 (1+|x|)^{[\gamma]+1}$ for all $x \in \mathbf{R}^n$. Thus, g can be viewed as an element of $\mathscr{S}'$, and one has $f_{L_m} - P_{L_m}^{[\gamma]} \to g$ in $\mathscr{S}'(\mathbf{R}^n)$. Since both g_1 and g_2 are in $\mathscr{C}^{[[\gamma]]}$, it follows that so is g.