Taking $\xi = N(x)$, this gives for any $x \in \mathbf{R}$

$$\Pi_{N(x)}(f)(x) = \lim_{\substack{K\to\infty \\ L\to\infty}} \frac{1}{2KL} \int_0^L \int_{-K}^K \int_0^1 G_{N(x),y,\eta,\lambda}(f)(x)\, d\lambda\, dy\, d\eta$$

and hence

$$|\Pi_{N(x)}(f)(x)| \le \liminf_{\substack{K\to\infty \\ L\to\infty}} \frac{1}{2KL} \int_0^L \int_{-K}^K \int_0^1 |G_{N(x),y,\eta,\lambda}(f)(x)|\, d\lambda\, dy\, d\eta\,.$$

We now apply the $L^{2,\infty}$ quasi-norm on both sides and we use Fatou's lemma for weak L^2; see Exercise 1.1.12(d) in [156]. Since modulations, translations, and L^2-dilations are isometries on L^2, we reduce the sought estimate for the operator in (6.1.25) to the corresponding estimate for $f \mapsto A_{N(x)}(f)(x) = \mathfrak{D}_N(f)(x)$.

To justify certain algebraic manipulations we fix a finite subset $\mathbf{P}$ of $\mathbf{D}$ and we define

$$\mathfrak{D}_{N,\mathbf{P}}(f)(x) = \sum_{s\in\mathbf{P}} (\chi_{\omega_{s(2)}} \circ N)(x) \left\langle f \,|\, \varphi_s \right\rangle \varphi_s(x)\,. \tag{6.1.28}$$

To prove (6.1.27) it suffices to show that there exists a $C>0$ such that for all f in $\mathscr{S}(\mathbf{R})$, all finite subsets $\mathbf{P}$ of $\mathbf{D}$, and all real-valued measurable functions N on the line we have

$$\big\|\mathfrak{D}_{N,\mathbf{P}}(f)\big\|_{L^{2,\infty}} \le C\big\|f\big\|_{L^2}\,. \tag{6.1.29}$$

The important point is that the constant C in (6.1.29) is independent of f, $\mathbf{P}$, and the measurable function N. Once (6.1.29) is known, then taking a sequence of sets $\mathbf{P}_L \to \mathbf{D}$, as $L \to \infty$ and using the absolute convergence of the series, we obtain (6.1.27).

To prove (6.1.29) we use duality. In view of the result of Exercises 1.4.12(c) in [156], it suffices to prove that for all $f \in \mathscr{S}(\mathbf{R})$ we have

$$\left| \int_{\mathbf{R}} \mathfrak{D}_{N,\mathbf{P}}(f)\, g\, dx \right| = \left| \sum_{s\in\mathbf{P}} \left\langle (\chi_{\omega_{s(2)}} \circ N)\varphi_s, g \right\rangle \left\langle \varphi_s \,|\, f \right\rangle \right| \le C\big\|g\big\|_{L^{2,1}} \big\|f\big\|_{L^2}\,. \tag{6.1.30}$$

Using the result of Exercise 1.4.7 in [156], (6.1.30) will follow from the fact that for all measurable subsets E of the real line with finite measure we have

$$\left| \int_E \mathfrak{D}_{N,\mathbf{P}}(f)\, dx \right| = \left| \sum_{s\in\mathbf{P}} \left\langle (\chi_{\omega_{s(2)}} \circ N)\varphi_s, \chi_E \right\rangle \left\langle \varphi_s \,|\, f \right\rangle \right| \le C|E|^{\frac12} \big\|f\big\|_{L^2}\,. \tag{6.1.31}$$

We obtain estimate (6.1.31) as a consequence of

$$\sum_{s\in\mathbf{P}} \left| \left\langle (\chi_{\omega_{s(2)}} \circ N)\varphi_s, \chi_E \right\rangle \left\langle f \,|\, \varphi_s \right\rangle \right| \le C|E|^{\frac12} \big\|f\big\|_{L^2} \tag{6.1.32}$$

for all f in $\mathscr{S}(\mathbf{R})$, all measurable functions N, all measurable sets E of finite measure, and all finite subsets $\mathbf{P}$ of $\mathbf{D}$. We therefore concentrate on estimate (6.1.32).