where we used an earlier observation about s and s', the Cauchy–Schwarz inequality, and the fact that

$$\sum_{\substack{s'\in\mathbf{D}_m\\ \omega_{s'}=\omega_s}} \big|\langle \varphi_s \,|\, \varphi_{s'}\rangle\big| \le C \sum_{\substack{s'\in\mathbf{D}_m\\ \omega_{s'}=\omega_s}} \left(1+\frac{\operatorname{dist}\,(I_s,I_{s'})}{2^m}\right)^{-10} \le C_1\,,$$

which follows from the result in Appendix B.1. To estimate (6.1.10), we use that

$$\begin{aligned}\big|\langle f \,|\, \varphi_s\rangle\big| &\le C_2 \int_{\mathbf{R}} |f(y)|\,|I_s|^{-\frac12}\left(1+\frac{|y-c(I_s)|}{|I_s|}\right)^{-10} dy\\ &\le C_3\,|I_s|^{\frac12}\int_{\mathbf{R}}|f(y)|\left(1+\frac{|y-z|}{|I_s|}\right)^{-10}\frac{dy}{|I_s|}\\ &\le C_4\,|I_s|^{\frac12} M(f)(z),\end{aligned}$$

for all $z\in I_s$, in view of Theorem 2.1.10 in [156]. Since the preceding estimate holds for all $z\in I_s$, it follows that

$$\big|\langle f \,|\, \varphi_s\rangle\big|^2 \le (C_4)^2 |I_s| \inf_{z\in I_s} M(f)(z)^2 \le (C_4)^2 \int_{I_s} M(f)(x)^2\,dx. \tag{6.1.11}$$

Next we observe that the rectangles $s\in\mathbf{D}_m$ with the property that $\xi\in\omega_{s(2)}$ are all disjoint. This implies that the corresponding time intervals I_s are also disjoint. Thus, summing (6.1.11) over all $s\in\mathbf{D}_m$ with $\xi\in\omega_{s(2)}$, we obtain that

$$\begin{aligned}\sum_{s\in\mathbf{D}_m}\big|\langle f \,|\, \varphi_s\rangle\big|^2 \chi_{\omega_{s(2)}}(\xi) &\le (C_4)^2 \sum_{s\in\mathbf{D}_m}\chi_{\omega_{s(2)}}(\xi)\int_{I_s} M(f)(x)^2\,dx\\ &\le (C_4)^2\int_{\mathbf{R}} M(f)(x)^2\,dx,\end{aligned}$$

which establishes the required claim using the boundedness of the Hardy–Littlewood maximal operator M on $L^2(\mathbf{R})$. We conclude that each A_ξ^m, initially defined on $\mathscr{S}(\mathbf{R})$, admits an L^2-bounded extension and all these extensions have norms uniformly bounded in m and ξ. We denote these extensions also by A_ξ^m.

We now explain why $A_\xi=\sum_{m\in\mathbf{Z}} A_\xi^m$ is well defined on $L^2(\mathbf{R})$ and we examine its L^2 boundedness. For every fixed $\xi\in\mathbf{R}$ and each $m\in\mathbf{Z}$ there is at most one $\ell\in\mathbf{Z}$ such that the upper parts of the frequency components of the dyadic tiles

$$s=[k2^m,(k+1)2^m)\times[\ell 2^{-m},(\ell+1)2^{-m})\,,\qquad k\in\mathbf{Z}$$

contain ξ, i.e., they satisfy $(\ell+\frac12)2^{-m}\le\xi<(\ell+1)2^{-m}$. For a given $g\in L^2(\mathbf{R})$ let $g_m=0$ if no such ℓ exists. For those m for which ℓ exists define g_m via

$$\widehat{g_m}=\widehat{g}\,\chi_{[\ell 2^{-m},(\ell+\frac12)2^{-m})}$$

and notice that $A_\xi^m(g_m) = A_\xi^m(g)$. Moreover, $\widehat{g_m}$ and $\widehat{g_{m'}}$ have disjoint supports if $m \neq m'$. We use these observations to obtain

$$\begin{aligned}\sum_{m\in\mathbf{Z}} \big\|A_\xi^m(g)\big\|_{L^2}^2 &= \sum_{m\in\mathbf{Z}} \big\|A_\xi^m(g_m)\big\|_{L^2}^2 \\ &\le C_5 \sum_{m\in\mathbf{Z}} \big\|g_m\big\|_{L^2}^2 \\ &= C_5 \sum_{m\in\mathbf{Z}} \big\|\widehat{g_m}\big\|_{L^2}^2 \\ &\le C_5 \|g\|_{L^2}^2 < \infty .\end{aligned} \tag{6.1.12}$$

As already observed, the supports of the Fourier transforms of $A_\xi^m(g)$ are pairwise disjoint when $m \in \mathbf{Z}$. This implies that $\big\langle A_\xi^m(g)\,|\,A_\xi^{m'}(g)\big\rangle = 0$ whenever $m \neq m'$. Consequently, given $\varepsilon > 0$ there is an N_0 such that for $M > N \geq N_0$ we have

$$\Big\|\sum_{N\le|m|\le M} A_\xi^m(g)\Big\|_{L^2}^2 = \sum_{N\le|m|\le M} \big\|A_\xi^m(g)\big\|_{L^2}^2 < \varepsilon^2 . \tag{6.1.13}$$

Thus the series $\sum_{m\in\mathbf{Z}} A_\xi^m(g)$ is Cauchy and it converges to an element of $L^2(\mathbf{R})$ which we denote by $A_\xi(g)$. Combining (6.1.12) and (6.1.13) we obtain that A_ξ is bounded from $L^2(\mathbf{R})$ to itself with norm at most C_5.

We now address the last assertion about the absolute pointwise convergence of the series in (6.1.8) for all $x \in \mathbf{R}$ when $f \in L^1(\mathbf{R})$ and $\xi > 0$. For fixed $x \in \mathbf{R}$, $\xi > 0$, we pick $m_0 \in \mathbf{Z}$ such that $2^{-m_0-1} \le \xi < 2^{-m_0}$. We notice that for each $m \in \mathbf{Z}$ there is only one horizontal row of tiles of size $2^m \times 2^{-m}$ whose upper parts contain ξ and thus appearing in the sum in (6.1.8). Moreover, for all the tiles s that appear in the sum in (6.1.8), the size of ω_s cannot be bigger than 2^{-m_0} since the top part of ω_s contains ξ. Thus if $I_s = [2^m k, 2^m(k+1))$, we must have $m \geq m_0$. Combining these observations with the fact that $|\langle f\,|\,\varphi_s\rangle| \le \|f\|_{L^1}\|\varphi_s\|_{L^\infty}$, we estimate the sum of the absolute value of each term of the series in (6.1.8) by

$$C\,\|f\|_{L^1} \sum_{m\ge m_0}\sum_{k\in\mathbf{Z}} 2^{-\frac{m}{2}} \frac{2^{-\frac{m}{2}}}{(1+2^{-m}|x-2^m(k+\frac{1}{2})|)^2} \tag{6.1.14}$$

for some constant $C > 0$. Summing first over $k \in \mathbf{Z}$ and then over $m \geq m_0$, we obtain that the series in (6.1.8) converges absolutely for all $x \in \mathbf{R}$ and is bounded above by a constant multiple of $\xi\,\|f\|_{L^1}$. □

6.1.2 Discretization of the Carleson Operator

We let $h \in \mathscr{S}(\mathbf{R})$, $\xi \in \mathbf{R}$, and for each $m \in \mathbf{Z}$, $y, \eta \in \mathbf{R}$, and $\lambda \in [0,1]$ we introduce the operators

$$B^m_{\xi,y,\eta,\lambda}(h) = \sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))\,\big\langle D^{2^\lambda}\tau^y M^\eta(h)\,|\,\varphi_s\big\rangle M^{-\eta}\tau^{-y}D^{2^{-\lambda}}(\varphi_s)\,.$$