it suffices to obtain $L^2 \to L^{2,\infty}$ bounds for the *one-sided maximal operators*

$$\mathscr{C}_1(f)(x) = \sup_{N>0} \left| \int_{-\infty}^{N} \widehat{f}(\xi) e^{2\pi i x \xi} \, d\xi \right|,$$
$$\mathscr{C}_2(f)(x) = \sup_{N>0} \left| \int_{-\infty}^{-N} \widehat{f}(\xi) e^{2\pi i x \xi} \, d\xi \right|,$$

acting on a Schwartz function f (with bounds independent of f). Note that

$$\mathscr{C}_2(f)(x) \le |f(x)| + \mathscr{C}_1(\widetilde{f})(-x),$$

where $\widetilde{f}(x) = f(-x)$ is the usual reflection operator. Therefore, it suffices to obtain bounds only for $\mathscr{C}_1$.

For $a > 0$ and $y \in \mathbf{R}$ we define the translation operator τ^y, the modulation operator M^a, and the dilation operator D^a as follows:

$$\tau^y(f)(x) = f(x-y),$$
$$D^a(f)(x) = a^{-\frac{1}{2}} f(a^{-1}x),$$
$$M^y(f)(x) = f(x) e^{2\pi i y x}.$$

These operators are isometries on $L^2(\mathbf{R})$.

We break down the proof of Theorem 6.1.1 into several steps.

6.1.1 Preliminaries

We denote rectangles of area 1 in the (x,ξ) plane by s, t, u, etc. All rectangles considered in the sequel have sides parallel to the axes. We think of x as the time coordinate and of ξ as the frequency coordinate. For this reason we refer to the (x,ξ) coordinate plane as the time–frequency plane. The projection of a rectangle s on the time axis is denoted by I_s, while its projection on the frequency axis is denoted by ω_s. Thus a rectangle s is just $s = I_s \times \omega_s$. Rectangles with sides parallel to the axes and area equal to one are called *tiles*.

The center of an interval I is denoted by $c(I)$. Also for $a > 0$, aI denotes an interval with the same center as I whose length is $a|I|$. Given a tile s, we denote by $s(1)$ its bottom half and by $s(2)$ its upper half defined by

$$s(1) = I_s \times \big(\omega_s \cap (-\infty, c(\omega_s))\big), \qquad s(2) = I_s \times \big(\omega_s \cap [c(\omega_s), +\infty)\big).$$

These sets are called *semitiles*. The projections of these sets on the frequency axes are denoted by $\omega_{s(1)}$ and $\omega_{s(2)}$, respectively. See Figure 6.1.

A dyadic interval is an interval of the form $[m2^k, (m+1)2^k)$, where k and m are integers. We denote by $\mathbf{D}$ the set of all rectangles $I \times \omega$ with I, ω dyadic intervals and $|I|\,|\omega| = 1$. Such rectangles are called *dyadic tiles*. ~~We denote by **D** the set of~~

~~all dyadic tiles.~~ For every integer m, we denote by $\mathbf{D}_m$ the set of all tiles $s \in \mathbf{D}$ such that $|I_s| = 2^m$. We call these dyadic tiles *of scale m*.

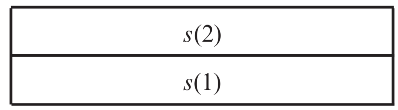

Fig. 6.1 The lower and the upper parts of a tile s.

We fix a Schwartz function φ such that $\widehat{\varphi}$ takes values in $[0,1]$ and supported in the interval $[-1/10, 1/10]$, and equal to 1 on the interval $[-9/100, 9/100]$. For each tile s, we introduce a function φ_s as follows:

$$\varphi_s(x) = |I_s|^{-\frac{1}{2}} \varphi\left(\frac{x - c(I_s)}{|I_s|}\right) e^{2\pi i c(\omega_{s(1)})x}. \tag{6.1.4}$$

This function is localized in frequency near $c(\omega_{s(1)})$. Using the previous notation, we have

$$\varphi_s = M^{c(\omega_{s(1)})} \tau^{c(I_s)} D^{|I_s|}(\varphi).$$

Observe that

$$\widehat{\varphi_s}(\xi) = |\omega_s|^{-\frac{1}{2}} \widehat{\varphi}\left(\frac{\xi - c(\omega_{s(1)})}{|\omega_s|}\right) e^{2\pi i (c(\omega_{s(1)}) - \xi) c(I_s)}, \tag{6.1.5}$$

from which it follows that $\widehat{\varphi_s}$ is supported in $\frac{2}{5}\omega_{s(1)}$. Also observe that the functions φ_s have the same $L^2(\mathbf{R})$ norm.

Recall the complex inner product notation for $f, g \in L^2(\mathbf{R})$:

$$\langle f \,|\, g \rangle = \int_{\mathbf{R}} f(x)\overline{g(x)}\, dx. \tag{6.1.6}$$

Given a real number ξ and $m \in \mathbf{Z}$, we introduce an operator

$$A_\xi^m(f) = \sum_{s \in \mathbf{D}_m} \chi_{\omega_{s(2)}}(\xi) \langle f \,|\, \varphi_s \rangle \varphi_s, \tag{6.1.7}$$

for functions $f \in \mathscr{S}(\mathbf{R})$. The series in (6.1.7) converges absolutely and in L^2 for f in the Schwartz class (see Exercise 6.1.9) and thus A_ξ^m is well defined on $\mathscr{S}(\mathbf{R})$. Note that for a fixed m, the sum in (6.1.7) is taken over the row of dyadic rectangles of size $2^m \times 2^{-m}$ whose tops contain the horizontal line at height ξ. The Fourier transforms of the operators A_ξ^m are supported in a horizontal strip contained in $(-\infty, \xi]$ of width $\frac{2}{5}2^{-m}$. Notice that if the characteristic function were missing in (6.1.7), then for a suitable function φ, the sum would be equal to a multiple of $f(x)$; cf. Exercise 6.1.9. Thus for each $m \in \mathbf{Z}$ the operator $A_\xi^m(f)$ may be viewed as a "piece" of the multiplier operator $f \mapsto \big(\widehat{f}\chi_{(-\infty,\xi]}\big)^\vee$. Summing over m yields a better approximation to this half-line multiplier operator.