we show that $\partial_j(|\xi|^z\widehat{\varphi}(\xi))(0)$ exists. Since every Taylor polynomial of $\widehat{\varphi}$ at zero is identically equal to zero, it follows from Taylor's theorem that $|\widehat{\varphi}(\xi)| \le C_M|\xi|^M$ for every $M \in \mathbf{Z}^+$, whenever ξ lies in a compact set. Consequently, if $M > 1 - \operatorname{Re} z$,

$$\frac{|te_j|^z\widehat{\varphi}(te_j)}{t}$$

tends to zero as $t \to 0$ when e_j is the vector with 1 in the jth entry and zero elsewhere. This shows that all partials of $|\xi|^z\widehat{\varphi}(\xi)$ at zero exist and are equal to zero. By induction we assume that $\partial^\gamma(|\xi|^z\widehat{\varphi}(\xi))(0) = 0$, and we need to prove that $\partial_j\partial^\gamma(|\xi|^z\widehat{\varphi}(\xi))(0)$ also exists and equals zero. Applying Leibniz's rule, we express $\partial^\gamma(|\xi|^z\widehat{\varphi}(\xi))$ as a finite sum of derivatives of $|\xi|^z$ times derivatives of $\widehat{\varphi}(\xi)$. But for each $|\beta| \le |\gamma|$ we have $|\partial^\beta(\widehat{\varphi})(\xi)| \le C_{M,\beta}|\xi|^M$ for all $M \in \mathbf{Z}^+$ whenever $|\xi| \le 1$. Picking $M > |\gamma| + 1 - \operatorname{Re} z$ and using the fact that $|\partial^{\gamma-\beta}(|\xi|^z)| \le C_\alpha|\xi|^{\operatorname{Re} z - |\gamma| + |\beta|}$, we deduce that $\partial_j\partial^\gamma(|\xi|^z\widehat{\varphi}(\xi))(0)$ also exists and equals zero.

We have now proved that if φ belongs to $\mathscr{S}_0(\mathbf{R}^n)$, then so does $\big(|\xi|^z\widehat{\varphi}(\xi)\big)^\vee$ for all $z \in \mathbf{C}$. This allows us to introduce the operation of multiplication by $|\xi|^z$ on the Fourier transforms of distributions modulo polynomials. This is described in the following definition.

Definition 1.1.4. Let $s \in \mathbf{C}$ and $u \in \mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$. We define another distribution $(|\xi|^s\widehat{u})^\vee$ in $u \in \mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ by setting for all φ in $\mathscr{S}_0(\mathbf{R}^n)$

$$\langle (|\cdot|^s\widehat{u})^\vee, \varphi\rangle = \langle u, (|\cdot|^s\varphi^\vee)\widehat{\ }\rangle\,.$$

This definition is consistent with the corresponding operations on functions and makes sense since, as observed, φ in $\mathscr{S}_0(\mathbf{R}^n)$ implies that $(|\cdot|^s\widehat{\varphi})^\vee$ also lies in $\mathscr{S}_0(\mathbf{R}^n)$, and thus the action of u on this function is defined.

The next proposition allows us to deduce that an infinite sum of $\mathscr{C}^s$ functions is also in $\mathscr{C}^s$ under certain circumstances.

Proposition 1.1.5. *Let $N \in \mathbf{Z}^+$. Suppose that $\{g_i\}_{i\in\mathbf{Z}}$ are functions in $\mathscr{C}^{|\alpha|}(\mathbf{R}^n)$ for all multi-indices α with $|\alpha| \le N$ and that $\sum_{i\in\mathbf{Z}}\|\partial^\alpha g_i\|_{L^\infty} < \infty$ for all $|\alpha| \le N$. Then the function $g = \sum_{i\in\mathbf{Z}} g_i$ is in $\mathscr{C}^{|\alpha|}(\mathbf{R}^n)$ and*

$$\partial^\alpha g = \sum_{i\in\mathbf{Z}}\partial^\alpha g_i$$

for all $|\alpha| \le N$.

Proof. Let e_j be the vector in $\mathbf{R}^n$ with 1 in the jth coordinate and zero in the remaining ones. For $h \in \mathbf{R}\setminus\{0\}$ we have

$$\frac{g(x+he_j)-g(x)}{h} = \sum_{i\in\mathbf{Z}}\frac{g_i(x+he_j)-g_i(x)}{h}\,.$$