$$\begin{aligned}
&\leq C\delta^{\frac{3}{2}}\Big[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^2}^4\Big]\log(\delta^{-\frac{1}{2}})\\
&\leq C\delta^{\frac{3}{2}}(\delta^{\frac{3}{2}})^{(\frac{1}{2}-\frac{1}{4})4}\Big[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^4}^4\Big]\log\tfrac{1}{\delta}\\
&\leq C\delta^3\big(\log\tfrac{1}{\delta}\big)\big\|f\big\|_{L^4}^4\,.
\end{aligned}$$

□

We now prove Lemma 5.4.8, which we had left open.

Proof. The proof is based on interpolation. For fixed $\ell,\ell'\in I$ we define the bilinear operator

$$T_{\ell,\ell'}(g,h)=(g\chi_\ell^\delta)*(h\chi_{\ell'}^\delta)\,.$$

As we have previously observed, it is a simple geometric fact that the support of χ_ℓ^δ is contained in a rectangle of side length $\approx\delta$ in the direction $e^{2\pi i\delta^{1/2}\ell}$ and of side length $\approx\delta^{\frac{1}{2}}$ in the direction $ie^{2\pi i\delta^{1/2}\ell}$. Any two rectangles with these dimensions in the aforementioned directions have an intersection that depends on the angle between them. Indeed, if $\ell\neq\ell'$, this intersection is contained in a rhombus of side $\delta/\sin(2\pi\delta^{\frac{1}{2}}|\ell-\ell'|)$ and height δ, and hence the measure of the intersection is ~~seen easily to be~~ at most a constant multiple of

$$\delta\cdot\frac{\delta}{\sin(2\pi\delta^{\frac{1}{2}}|\ell-\ell'|)}\,.$$

As for ℓ,ℓ' in the index set I we have $2\pi\delta^{\frac{1}{2}}|\ell-\ell'|<\pi/4$, the sine is comparable to its argument, and we conclude that the measure of the intersection is at most

$$C\,\delta^{\frac{3}{2}}(1+|\ell-\ell'|)^{-1}\,.$$

It follows that

$$\big\|\chi_\ell^\delta*\chi_{\ell'}^\delta\big\|_{L^\infty}=\sup_{z\in\mathbf{R}^2}|(z-\operatorname{supp}(\chi_\ell^\delta))\cap\operatorname{supp}(\chi_{\ell'}^\delta)|\leq\frac{C\delta^{\frac{3}{2}}}{1+|\ell-\ell'|}\,,$$

which implies the estimate

$$\begin{aligned}
\big\|T_{\ell,\ell'}(g,h)\big\|_{L^\infty}&\leq\big\|\chi_\ell^\delta*\chi_{\ell'}^\delta\big\|_{L^\infty}\big\|g\big\|_{L^\infty}\big\|h\big\|_{L^\infty}\\
&\leq\frac{C\delta^{\frac{3}{2}}}{1+|\ell-\ell'|}\big\|g\big\|_{L^\infty}\big\|h\big\|_{L^\infty}\,.
\end{aligned}\tag{5.4.39}$$

Also, the estimate

$$\big\|T_{\ell,\ell'}(g,h)\big\|_{L^1}\leq\big\|g\chi_\ell^\delta\big\|_{L^1}\big\|h\chi_{\ell'}^\delta\big\|_{L^1}\leq\|g\|_{L^1}\|h\|_{L^1}\tag{5.4.40}$$