show that there exist positive constants C,δ (depending only on n and $\operatorname{Re}\lambda$) such that for all functions f in $L^p(\mathbf{R}^n)$ we have

$$\big\|T_j^\lambda(f)\big\|_{L^p(\mathbf{R}^n)} \le C\, e^{c_0|\operatorname{Im}\lambda|^2} 2^{-j\delta}\big\|f\big\|_{L^p(\mathbf{R}^n)}. \tag{5.4.15}$$

Once (5.4.15) is established, the L^p boundedness of the operator $f\mapsto K^\lambda * f$ follows by summing the series in (5.4.14).

As a consequence of (5.4.13) we obtain that

$$\begin{aligned} |K_j^\lambda(x)| \;&\le\; C_3(\operatorname{Re}\lambda)\, e^{c_0|\operatorname{Im}\lambda|^2}\,(1+|x|)^{-\frac{n+1}{2}-\operatorname{Re}\lambda}|\psi_j(x)| \\ &\le\; C' 2^{-(\frac{n+1}{2}+\operatorname{Re}\lambda)j}, \end{aligned} \tag{5.4.16}$$

since $\psi_j(x)=\psi(2^{-j}x)$ and ψ is supported in the annulus $\frac{1}{2}\le |x|\le 2$. From this point on, we tacitly assume that the constants containing a prime grow at most exponentially in $|\operatorname{Im}\lambda|^2$. Since K_j^λ is supported in a ball of radius 2^{j+1} and satisfies (5.4.16), we deduce the estimate

$$\big\|\widehat{K_j^\lambda}\big\|_{L^2}^2 = \big\|K_j^\lambda\big\|_{L^2}^2 \le C'' 2^{-(n+1+2\operatorname{Re}\lambda)j} 2^{nj} = C'' 2^{-(1+2\operatorname{Re}\lambda)j}. \tag{5.4.17}$$

We need another estimate for $\widehat{K_j^\lambda}$. We claim that for all $M\ge n+1$ there is a constant $C_{M,n,\beta}$ such that

$$\int_{|\xi|\le\frac{1}{8}} |\widehat{K_j^\lambda}(\xi)|^2|\xi|^{-\beta}\,d\xi \le C_{M,n,\beta}\, 2^{-2j(M-n)}, \quad \beta<n. \tag{5.4.18}$$

Indeed, since $\widehat{K^\lambda}(\xi)$ is supported in $|\xi|\ge\frac{1}{2}$ [recall that the function η was chosen equal to 1 on $B(0,\frac{1}{2})$], we have

$$|\widehat{K_j^\lambda}(\xi)| = |(\widehat{K^\lambda} * \widehat{\psi_j})(\xi)| \le 2^{jn}\int_{\frac{1}{2}\le|\xi-\omega|\le 1} (1-|\xi-\omega|^2)_+^{\operatorname{Re}\lambda}|\widehat{\psi}(2^j\omega)|\,d\omega.$$

Suppose that $|\xi|\le\frac{1}{8}$. Since $|\xi-\omega|\ge\frac{1}{2}$, we must have $|\omega|\ge\frac{3}{8}$. Then

$$|\widehat{\psi}(2^j\omega)| \le C_M(2^j|\omega|)^{-M} \le (8/3)^M C_M 2^{-jM},$$

from which it follows easily that

$$\sup_{|\xi|\le\frac{1}{8}} |\widehat{K_j^\lambda}(\xi)| \le C_M' 2^{-j(M-n)}. \tag{5.4.19}$$

Then (5.4.18) is a consequence of (5.4.19) and of the fact that the function $|\xi|^{-\beta}$ is integrable near the origin.

We now return to estimate (5.4.15). A localization argument (Exercise 5.4.4) allows us to reduce estimate (5.4.15) to functions f that are supported in a cube of