where

$$\beta = 1-(n-1)(\frac{1}{p}-\frac{1}{2})$$

and I_β is the Riesz potential (or fractional integral) given in Definition 1.2.1. Using Theorem 1.2.3 with $s=\beta$, $n=1$, and $q=p'$, we obtain that the last displayed equation is bounded by a constant multiple of

$$\Big\| \big\| f(\,\cdot\,,t)\big\|_{L^p(\mathbf{R}^{n-1})}\Big\|_{L^p(\mathbf{R},dt)} = \big\|f\big\|_{L^p(\mathbf{R}^n)} .$$

The condition $\frac{1}{p}-\frac{1}{q}=\frac{s}{n}$ on the indices p,q,s,n assumed in Theorem 1.2.3 translates exactly to

$$\frac{1}{p}-\frac{1}{p'}=\frac{\beta}{1}=1-\frac{n-1}{p}+\frac{n-1}{2},$$

which is equivalent to $p=\frac{2(n+1)}{n+3}$. This concludes the proof of estimate (5.4.7) in which the measure $\sigma^\vee$ is replaced by $(\varphi_j\,d\sigma)^\vee$. Estimates for the remaining $(\varphi_j\,d\sigma)^\vee$ follow by a similar argument in which the role of the last coordinate is played by some other coordinate. The final estimate (5.4.7) follows by summing j over the finite set F. The proof of the theorem is now complete. $\square$

5.4.3 Applications to Bochner–Riesz Multipliers

We now apply the restriction theorem obtained in the previous subsection to the Bochner–Riesz problem. In this subsection we prove the following result.

Theorem 5.4.6. *For* $\mathrm{Re}\,\lambda > \frac{n-1}{2(n+1)}$, *the Bochner–Riesz operator* B^λ *is bounded on* $L^p(\mathbf{R}^n)$ *for p in the optimal range*

$$\frac{2n}{n+1+2\mathrm{Re}\,\lambda} < p < \frac{2n}{n-1-2\mathrm{Re}\,\lambda} .$$

Proof. The proof is based on the following two estimates:

$$\big\|B^\lambda\big\|_{L^1(\mathbf{R}^n)\to L^1(\mathbf{R}^n)} \le C_1(\mathrm{Re}\,\lambda)\, e^{c_0|\mathrm{Im}\,\lambda|^2} \qquad \text{when } \mathrm{Re}\,\lambda > \tfrac{n-1}{2}, \tag{5.4.11}$$

$$\big\|B^\lambda\big\|_{L^p(\mathbf{R}^n)\to L^p(\mathbf{R}^n)} \le C_2(\mathrm{Re}\,\lambda)\, e^{c_0|\mathrm{Im}\,\lambda|^2} \qquad \text{when } \mathrm{Re}\,\lambda > \tfrac{n-1}{2(n+1)}, \tag{5.4.12}$$

where $p=\frac{2(n+1)}{n+3}$ and C_1, C_2 are constants that depend on n and $\mathrm{Re}\,\lambda$, while c_0 is an absolute constant. Once (5.4.11) and (5.4.12) are known, the required conclusion is a consequence of Theorem 1.3.7 in [156]. Recall that B^λ is given by convolution with the kernel K_λ defined in (5.2.1). This kernel satisfies

$$|K_\lambda(x)| \le C_3(\mathrm{Re}\,\lambda)\, e^{c_0|\mathrm{Im}\,\lambda|^2}(1+|x|)^{-\frac{n+1}{2}-\mathrm{Re}\,\lambda} \tag{5.4.13}$$