We now define the homogeneous Lipschitz spaces.

Definition 1.4.3. For $\gamma > 0$ we define

$$\|f\|_{\dot{\Lambda}_\gamma} = \sup_{x\in\mathbf{R}^n} \sup_{h\in\mathbf{R}^n\setminus\{0\}} \frac{|D_h^{[\gamma]+1}(f)(x)|}{|h|^\gamma}$$

and we let $\dot{\Lambda}_\gamma$ be the space of all continuous functions f on $\mathbf{R}^n$ that satisfy $\|f\|_{\dot{\Lambda}_\gamma} < \infty$. We call $\dot{\Lambda}_\gamma$ the *homogeneous Lipschitz space* of order γ.

We verify that elements of $\dot{\Lambda}_\gamma$ have at most polynomial growth at infinity. Indeed, identity (1.4.2) implies for all $h \in \mathbf{R}^n$

$$D_h^{k+1}(f-f(0))(0) = \sum_{s=1}^{k+1}(-1)^{k+1-s}\binom{k+1}{s}(f(sh)-f(0))$$

and thus

$$\begin{aligned}|f((k+1)h)-f(0)| &\le \sum_{s=1}^{k}\binom{k+1}{s}|f(sh)-f(0)| + \|f\|_{\dot{\Lambda}_\gamma}|h|^{k+1} \\ &\le 2^{k+1}\big[\sup_{s\in\{1,\dots,k\}}|f(sh)-f(0)| + \|f\|_{\dot{\Lambda}_\gamma}|h|^{k+1}\big].\end{aligned}$$

Replacing h by $(k+1)h$, we obtain for all $h\in\mathbf{R}^n$

$$\begin{aligned}|f((k+1)^2h)-f(0)| &\le 2^{k+1}\big[\sup_{s\in\{1,\dots,k\}}|f(s(k+1)h)-f(0)| + \|f\|_{\dot{\Lambda}_\gamma}|(k+1)h|^{k+1}\big] \\ &\le 2^{k+1}\big[2^{k+1}\sup_{s,s'\in\{1,\dots,k\}}|f(ss'h)-f(0)| + \|f\|_{\dot{\Lambda}_\gamma}|(k+1)h|^{k+1}\big] \\ &\le (2^{k+1})^2\big[\sup_{s\in\{1,\dots,k^2\}}|f(sh)-f(0)| + \|f\|_{\dot{\Lambda}_\gamma}|(k+1)h|^{k+1}\big],\end{aligned}$$

and thus continuing in this way for all $M\in\mathbf{Z}^+$ and $h\in\mathbf{R}^n$ we deduce

$$|f((k+1)^Mh)-f(0)| \le (2^{k+1})^M\big[\sup_{s\in\{1,\dots,k^M\}}|f(sh)-f(0)| + \|f\|_{\dot{\Lambda}_\gamma}|(k+1)^{M-1}h|^{k+1}\big].$$

It follows from this that

$$|f(h)-f(0)| \le (2^{k+1})^M\left[\sup_{s\in\{1,\dots,k^M\}}|f(s(k+1)^{-M}h)-f(0)| + \|f\|_{\dot{\Lambda}_\gamma}\frac{|h|^{k+1}}{(k+1)^{(k+1)}}\right].$$

Given $|h|>1$, there is an $M\in\mathbf{Z}^+$ such that $(\frac{k+1}{k})^{M-1} < |h| \le (\frac{k+1}{k})^M$. Then, if $c(k) = (k+1)/\log_2(\frac{k+1}{k})$, we have

$$(2^{k+1})^M = (\tfrac{k+1}{k})^{Mc(k)} \le (\tfrac{k+1}{k})^{c(k)}|h|^{c(k)}.$$

But f is continuous, so $\|f\|_{L^\infty(\overline{B(0,1)})} < \infty$, and consequently for all $|h| > 1$ we obtain

$$|f(h) - f(0)| \le \left(\tfrac{k+1}{k}\right)^{c(k)} \left[2\|f\|_{L^\infty(\overline{B(0,1)})} + \|f\|_{\dot{\Lambda}_\gamma}\right] |h|^{c(k)}.$$

We conclude that functions in $\dot{\Lambda}_\gamma$ have at most polynomial growth at infinity and they can be thought of as elements of $\mathscr{S}'(\mathbf{R}^n)$.

Since elements of $\dot{\Lambda}_\gamma$ can be viewed as tempered distributions, we extend the definition of $D_h^k(u)$ to tempered distributions. For $u \in \mathscr{S}'(\mathbf{R}^n)$ we define another tempered distribution $D_h^k(u)$ via the identity

$$\left\langle D_h^k(u), \varphi \right\rangle = \left\langle u, D_{-h}^k(\varphi) \right\rangle$$

for all φ in the Schwartz class.

Constant functions f satisfy $D_h(f)(x) = 0$ for all $h, x \in \mathbf{R}^n$, and therefore the quantity $\|\cdot\|_{\dot{\Lambda}_\gamma}$ is insensitive to constants. Similarly, the expressions $D_h^{[\gamma]+1}(f)$ and $\|f\|_{\dot{\Lambda}_\gamma}$ do not recognize polynomials of degree up to $[\gamma]$. Moreover, polynomials are the only continuous functions with this property; see Exercise 1.4.1. This means that the quantity $\|\cdot\|_{\dot{\Lambda}_\gamma}$ is not a norm but only a seminorm. It can be made a norm if we consider equivalent classes of functions modulo polynomials. For this reason we often view $\dot{\Lambda}_\gamma$ as a subspace of $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}_{[\gamma]}(\mathbf{R}^n)$, where $\mathscr{P}_d$ is the space of polynomials of degree at most d for $d \ge 0$.

Examples 1.4.4. Let $a \in \mathbf{R}^n$, and let $0 < \gamma < 1$. Then the function $h(x) = \cos(x \cdot a)$ lies in $\Lambda_\gamma(\mathbf{R}^n)$ since $|h(x) - h(y)| \le \min(2, |a|\,|x-y|)$, and thus

$$|h(x) - h(y)| \le 2^{1-\gamma}|a|^\gamma |x-y|^\gamma.$$

Also, the function $x \mapsto |x|^\gamma$ lies in $\dot{\Lambda}_\gamma(\mathbf{R}^n)$ since $\big||x+h|^\gamma - |x|^\gamma\big| \le |h|^\gamma$ for $0 < \gamma < 1$.

Interesting examples of functions in Lipschitz spaces of higher order arise by the powers of the absolute value. Consider, for instance, the function $|x|^2$ on $\mathbf{R}^n$: we have $D_h(|x|^2) = 2|h|^2$, and thus $|x|^2 \in \dot{\Lambda}_\gamma(\mathbf{R}^n)$ if and only if $\gamma \ge 2$.

Another example is given by the function $|x|^{3/2}$ on $\mathbf{R}^n$ which has continuous partial derivatives at any point: $\partial_j |x|^{3/2} = \frac{3}{2} x_j |x|^{-1/2}$, $j = 1, \dots, n$, on $\mathbf{R}^n$ (with a value of 0 at the origin), while $(|x|^{3/2})' = \frac{3}{2}|x|^{1/2}\operatorname{sgn} x$ when $n = 1$. We claim that the function $|x|^{3/2}$ lies in $\dot{\Lambda}_{3/2}(\mathbf{R}^n)$ and that the functions $x_j|x|^{-1/2}$ lie in $\dot{\Lambda}_{1/2}(\mathbf{R}^n)$. To verify these assertions, we first prove the inequality

$$\left| \frac{x_j + h_j}{|x+h|^{\frac{1}{2}}} - \frac{x_j}{|x|^{\frac{1}{2}}} \right| \le C\,|h|^{\frac{1}{2}} \tag{1.4.3}$$

by considering the following three cases: (a) $x = 0$ and $h \ne 0$, which is trivial; (b) $x \ne 0$ and $2|h| < |x|$, in which case both functions are smooth and the mean value theorem yields a bound of the form $c\,|h|\,|x+\xi|^{-1/2}$ for some $|\xi| \le |h|$, proving

(1.4.3), since $|x+\xi| \geq |x| - |\xi| \geq |x| - |h| \geq |h|$; and (c) $2|h| \geq |x|$ and $h \neq -x \neq 0$, in which case the left-hand side of (1.4.3) is bounded by

$$|x+h|^{1/2} + |x|^{1/2} \leq C|h|^{1/2}.$$

Now for some $\xi \in \mathbf{R}^n$, with $|\xi| \leq |h|$, we have[2]

$$D_h^2(|x|^{3/2}) = \nabla(|x|^{3/2})(x+h+\xi)\cdot h - \nabla(|x|^{3/2})(x+\xi)\cdot h$$

and applying (1.4.3) we deduce that

$$|D_h^2(|x|^{3/2})| \leq C\,|h|^{3/2}.$$

We will make use of the following properties of the difference operators D_h^k.

Proposition 1.4.5. *Let f be a $\mathscr{C}^m$ function on $\mathbf{R}^n$ for some $m \in \mathbf{Z}^+$. Then for all $h = (h_1, \dots, h_n)$ and $x \in \mathbf{R}^n$ the following identity holds:*

$$D_h(f)(x) = \int_0^1 \sum_{j=1}^n h_j\,(\partial_j f)(x+sh)\,ds\,. \tag{1.4.4}$$

More generally, we have that

$$D_h^m(f)(x) = \\ \int_0^1 \cdots \int_0^1 \sum_{j_1=1}^n \cdots \sum_{j_m=1}^n h_{j_1}\cdots h_{j_m}(\partial_{j_1}\cdots\partial_{j_m} f)(x+(s_1+\cdots+s_m)h)\,ds_1\cdots ds_m. \tag{1.4.5}$$

Consequently, if, for some $\gamma \in (0,1)$, $\partial^\alpha f$ lies in $\dot{\Lambda}_\gamma$ for all multi-indices $|\alpha| = m$, then f lies in $\dot{\Lambda}_{m+\gamma}$.

Proof. Identity (1.4.4) is a consequence of the fundamental theorem of calculus applied to the function $t \mapsto f((1-t)x + t(x+h))$ on $[0,1]$, whereas identity (1.4.5) follows from (1.4.4) by induction.

Now suppose that $\partial^\alpha f$ lie in $\dot{\Lambda}_\gamma$ for all multi-indices $|\alpha| = m$. Apply D_h on both sides of (1.4.5); using that

$$|D_h(\partial_{j_1}\cdots\partial_{j_m} f)(x+(s_1+\cdots+s_m)h)| \leq \big\|\partial_{j_1}\cdots\partial_{j_m} f\big\|_{\dot{\Lambda}_\gamma}|h|^\gamma$$

we obtain

$$|D_h^{m+1}(f)(x)| \leq |h|^{m+\gamma}\sum_{j_1=1}^n \cdots \sum_{j_m=1}^n \big\|\partial_{j_1}\cdots\partial_{j_m} f\big\|_{\dot{\Lambda}_\gamma},$$

which proves that f lies in $\dot{\Lambda}_{m+\gamma}$. □

[2] We used that $g(b) - g(a) = \int_0^1 \nabla g\big((1-t)a + tb\big)\cdot(b-a)\,dt = \nabla g\big((1-t^*)a + t^*b\big)\cdot(b-a)$ for all $a, b \in \mathbf{R}^n$, for a $\mathscr{C}^1$ function g on $\mathbf{R}^n$ and some $t^* \in (0,1)$, depending on g, a, b.

1.4.2 Littlewood–Paley Characterization of Homogeneous Lipschitz Spaces

We now characterize the homogeneous Lipschitz spaces using the Littlewood–Paley operators Δ_j. As in the previous section, we fix a radial Schwartz function Ψ whose Fourier transform is nonnegative, is supported in the annulus $1-\frac{1}{7} \le |\xi| \le 2$, is equal to one on the annulus $1 \le |\xi| \le 2-\frac{2}{7}$, and satisfies

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1 \tag{1.4.6}$$

for all $\xi \neq 0$. The Littlewood–Paley operators Δ_j^Ψ associated with Ψ are given by multiplication on the Fourier transform side by the smooth bump $\widehat{\Psi}(2^{-j}\xi)$. Since a given f in $\dot{\Lambda}_\gamma$ has polynomial growth at infinity, it is a tempered distribution, and thus the convolution $\Psi_{2^{-j}} * f = \Delta_j^\Psi(f)$ is a well-defined smooth function of at most polynomial growth at infinity (cf. Theorem 2.3.20 in [156]). In the sequel we set $[[\gamma]] = [\gamma]$ if $\gamma \notin \mathbf{Z}^+$ and $[[\gamma]] = \gamma - 1$ if $\gamma \in \mathbf{Z}^+$ and we also set $\mathscr{C}^0 = \mathscr{C}$.

Theorem 1.4.6. *Let Ψ, Δ_j^Ψ be as above and $\gamma > 0$. Then there is a constant $C = C(n,\gamma,\Psi)$ such that for all f in $\dot{\Lambda}_\gamma$ we have the estimate*

$$\sup_{j\in\mathbf{Z}} 2^{j\gamma}\big\|\Delta_j^\Psi(f)\big\|_{L^\infty} \le C\big\|f\big\|_{\dot{\Lambda}_\gamma}. \tag{1.4.7}$$

Conversely, given f in $\mathscr{S}'(\mathbf{R}^n)$ satisfying

$$\sup_{j\in\mathbf{Z}} 2^{j\gamma}\big\|\Delta_j^\Psi(f)\big\|_{L^\infty} = C_0 < \infty, \tag{1.4.8}$$

there is a polynomial Q such that $|f(x)-Q(x)| \le C_{n,\gamma} C_0 (1+|x|)^{[\gamma]+1}$ for all $x \in \mathbf{R}^n$ and some constant $C_{n,\gamma}$. Moreover, $f-Q$ lies in $\mathscr{C}^{[[\gamma]]}(\mathbf{R}^n) \cap \dot{\Lambda}_\gamma(\mathbf{R}^n)$ and satisfies

$$\big\|f-Q\big\|_{\dot{\Lambda}_\gamma} \le C'(n,\gamma,\Psi)\,C_0 \tag{1.4.9}$$

for some constant $C'(n,\gamma,\Psi)$. In particular, functions in $\dot{\Lambda}_\gamma(\mathbf{R}^n)$ are in $\mathscr{C}^{[[\gamma]]}(\mathbf{R}^n)$.

Proof. We begin with the proof of (1.4.7). We first consider the case $0 < \gamma < 1$, which is very simple. Since each Δ_j^Ψ is given by convolution with a function with mean value zero, for a function $f \in \dot{\Lambda}_\gamma$ and every $x \in \mathbf{R}^n$ we write

$$\begin{aligned}
\Delta_j^\Psi(f)(x) &= \int_{\mathbf{R}^n} f(x-y)\Psi_{2^{-j}}(y)\,dy \\
&= \int_{\mathbf{R}^n} (f(x-y)-f(x))\Psi_{2^{-j}}(y)\,dy \\
&= 2^{-j\gamma}\int_{\mathbf{R}^n} \frac{D_{-y}(f)(x)}{|y|^\gamma}|2^jy|^\gamma 2^{jn}\Psi(2^jy)\,dy,
\end{aligned}$$