where the supremum is taken over all rectangles R in $\mathbf{R}^2$ of dimensions a and aN where $a>0$ is arbitrary. Here N is a fixed real number that is at least 10.

Example 5.3.2. Let $\Sigma=\{v\}$ consist of the vector $v=(a,b)$. Then with $v^\perp=(-b,a)$

$$\mathfrak{M}_\Sigma(f)(x)=\sup_{\substack{0<r\le 1\\ N>0}}\sup_{\substack{y \text{ with}\\ x-y\in R_{r,N}}}\frac{1}{rN^2}\iint_{y+R_{r,N}}|f(z)|\,dz,\quad R_{r,N}=\big\{tv+sv^\perp\colon\, |t|\le N,|s|\le rN\big\}.$$

If $\Sigma=\{(1,0),(0,1)\}$ consists of the two unit vectors along the axes, then

$$\mathfrak{M}_\Sigma=M_s\,,$$

where M_s is the strong maximal function defined in (5.3.2).

It is obvious that for each $\Sigma\subseteq\mathbf{S}^1$, the maximal function $\mathfrak{M}_\Sigma$ maps $L^\infty(\mathbf{R}^2)$ to itself with constant 1. But $\mathfrak{M}_\Sigma$ may not always be of weak type $(1,1)$, as the example M_s indicates; see Exercise 5.3.1. The boundedness of $\mathfrak{M}_\Sigma$ on $L^p(\mathbf{R}^2)$ in general depends on the set Σ.

An interesting case arises in the following example as well.

Example 5.3.3. For $N\in\mathbf{Z}^+$, let

$$\Sigma=\Sigma_N=\Big\{\big(\cos(\tfrac{2\pi j}{N}),\sin(\tfrac{2\pi j}{N})\big):\ j=0,1,2,\dots,N-1\Big\}$$

be the set of N uniformly spread directions on the circle. Then we expect $\mathfrak{M}_{\Sigma_N}$ to be L^p bounded with constant depending on N. There is a connection between the operator $\mathfrak{M}_{\Sigma_N}$ previously defined and the Kakeya maximal operator $\mathscr{K}_N$ defined in (5.2.21). In fact, Exercise 5.3.3 says that

$$\mathscr{K}_N(f)\le 20\,\mathfrak{M}_{\Sigma_N}(f) \tag{5.3.4}$$

for all locally integrable functions f on $\mathbf{R}^2$.

We now indicate why the norms of $\mathscr{K}_N$ and $\mathfrak{M}_{\Sigma_N}$ on $L^2(\mathbf{R}^2)$ grow as $N\to\infty$. We refer to Exercises 5.3.4 and 5.3.7 for the corresponding result for $p\neq 2$.

Proposition 5.3.4. *There is a constant c such that for any $N\ge 10$ we have*

$$\big\|\mathscr{K}_N\big\|_{L^2(\mathbf{R}^2)\to L^2(\mathbf{R}^2)}\ge c\log N \tag{5.3.5}$$

and

$$\big\|\mathscr{K}_N\big\|_{L^2(\mathbf{R}^2)\to L^{2,\infty}(\mathbf{R}^2)}\ge c\,(\log N)^{\frac12}\,. \tag{5.3.6}$$

Therefore, a similar conclusion follows for $\mathfrak{M}_{\Sigma_N}$.

Proof. We consider the family of functions $f_N(x)=\frac{1}{|x|}\chi_{3\le|x|\le N}$ defined on $\mathbf{R}^2$ for $N\ge 10$. Then we have

$$\big\|f_N\big\|_{L^2(\mathbf{R}^2)}\le c_1(\log N)^{\frac12}\,. \tag{5.3.7}$$