where the last equality is just Plancherel's identity on $I_0 = [-\frac{1}{2}, \frac{1}{2}]$. In view of the last identity, it suffices to analyze the operator given by convolution with the family of kernels k_t. By the Poisson summation formula (Theorem 3.2.8 in [156]) applied to the function $x \mapsto \psi(x)e^{2\pi i x t}$, we obtain

$$\begin{aligned} m_t(\xi) &= e^{-2\pi i \xi t} \sum_{j\in\mathbf{Z}} \psi(\xi - j) e^{2\pi i(\xi - j)t} \\ &= \sum_{j\in\mathbf{Z}} \big(\psi(\cdot)e^{2\pi i(\cdot)t}\big)\widehat{\ }(j)\, e^{2\pi i j\xi} e^{-2\pi i \xi t} \\ &= \sum_{j\in\mathbf{Z}} e^{2\pi i(j-t)\xi}\, \widehat{\psi}(j-t)\,. \end{aligned}$$

Taking inverse Fourier transforms, we obtain

$$k_t = \sum_{j\in\mathbf{Z}} \widehat{\psi}(j-t)\delta_{-j+t}\,,$$

where δ_b denotes Dirac mass at the point b. Therefore, k_t is a sum of Dirac masses with rapidly decaying coefficients. Since each Dirac mass has ~~Borel norm at most~~ total variation equal to 1, we conclude that for some constant C we have

$$\big\|k_t\big\|_{\mathscr{M}} \le \sum_{j\in\mathbf{Z}} |\widehat{\psi}(j-t)| \le C \sum_{j\in\mathbf{Z}} (1+|j-t|)^{-10} \le c_0\,, \tag{5.2.26}$$

where c_0 is independent of t. This says that the measures k_t have uniformly bounded norms. Take now $f \in L^p(\mathbf{R})$ and $p \ge 2$. Using identity (5.2.24), we obtain

$$\begin{aligned} \int_{\mathbf{R}} \Big(\sum_{j\in\mathbf{Z}} |P_j(f)(x)|^2\Big)^{\frac{p}{2}} dx &= \int_{\mathbf{R}} \Big(\sum_{j\in\mathbf{Z}} |P_j S_j(f)(x)|^2\Big)^{\frac{p}{2}} dx \\ &\le c_p \int_{\mathbf{R}} \Big(\sum_{j\in\mathbf{Z}} |S_j(f)(x)|^2\Big)^{\frac{p}{2}} dx\,, \end{aligned}$$

and the last inequality follows from Exercise 5.6.1(a) in [156]. The constant c_p depends only on p. Recalling identity (5.2.25), we write

$$\begin{aligned} c_p \int_{\mathbf{R}} \Big(\sum_{j\in\mathbf{Z}} |S_j(f)(x)|^2\Big)^{\frac{p}{2}} dx &\le c_p \int_{\mathbf{R}} \bigg(\int_{I_0} |(k_t * f)(x)|^2\, dt\bigg)^{\frac{p}{2}} dx \\ &\le c_p \int_{\mathbf{R}} \bigg(\int_{I_0} |(k_t * f)(x)|^p dt\bigg)^{\frac{p}{p}} dx \\ &= c_p \int_{I_0} \int_{\mathbf{R}} |(k_t * f)(x)|^p\, dx\, dt \\ &\le c_0\, c_p \int_{I_0} \int_{\mathbf{R}} |f(x)|^p\, dx\, dt \\ &= c_0\, c_p \big\|f\big\|_{L^p}^p\,, \end{aligned}$$