We define a rectangle R_j inside the angle $\angle A_jC_jB_j$ as in Figure 5.5. The rectangle R_j is defined so that one of its vertices is either A_j or B_j and the length of its longest side is $3\log(k+2)$.

We now make some calculations. Similar triangles show that the distance from C_0 to C_{2^k-1} is h_k-1. As a consequence we obtain that the longest possible length that either A_jC_j or B_jC_j can achieve is $\sqrt{5}h_k/2$. By symmetry we may assume that the length of A_jC_j is larger than that of B_jC_j as in Figure 5.5. We now have that

$$\frac{\sqrt{5}}{2}h_k < \frac{3}{2}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k+1}\Big) < \frac{3}{2}\big(1+\log(k+1)\big) < 3\log(k+2),$$

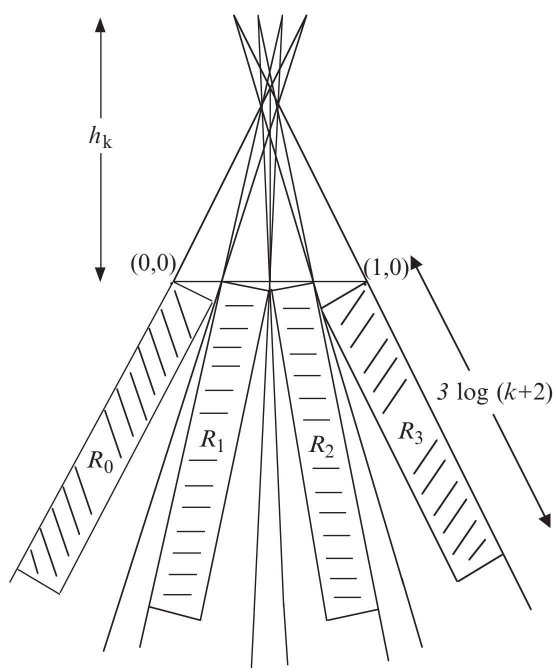
since $k \geq 1$ and $e < 3$. Hence R'_j contains the triangle $A_jB_jC_j$. We also have that

$$h_k = 1+\frac{1}{2}+\cdots+\frac{1}{k+1} > \log(k+2).$$

Using these two facts, we obtain

$$|R'_j \cap E| \geq \text{Area}(A_jB_jC_j) = \frac{1}{2}2^{-k}h_k > 2^{-k-1}\log(k+2). \tag{5.1.2}$$

Fig. 5.6 The rectangles R_j.



Denote by $|XY|$ the length of the line segment through the points X and Y. The law of sines applied to the triangle $A_jB_jD_j$ gives

$$|A_jD_j| = 2^{-k}\frac{\sin(\angle A_jB_jD_j)}{\sin(\angle A_jD_jB_j)} = 2^{-k}\frac{\sin(\angle A_jB_jD_j)}{\cos(\angle A_jC_jB_j)} \le \frac{2^{-k}}{\cos(\angle A_jC_jB_j)}. \tag{5.1.3}$$

But the law of cosines applied to the triangle $A_jB_jC_j$, combined with the facts $h_k \le |A_jC_j|, |B_jC_j| \le \sqrt{5}h_k/2$, and $h_k > \log(k+2) > 2^{-k+1}$ for $k \ge 1$, yields that

$$\cos(\angle A_jC_jB_j) = \frac{|A_jC_j|^2 + |B_jC_j|^2 - |A_jB_j|^2}{2|A_jC_j|\,|B_jC_j|} \ge \frac{h_k^2 + h_k^2 - (2^{-k})^2}{2\,\frac{5}{4}\,h_k^2} \ge \frac{4}{5} - \frac{2}{5}\cdot\frac{1}{4} \ge \frac{1}{2}.$$

Combining this estimate with (5.1.3) we obtain

$$|A_jD_j| \le 2^{-k+1} = 2\,|A_jB_j|.$$

Using this fact and (5.1.2), we deduce

$$|R_j' \cap E| \ge 2^{-k-1}\log(k+2) = \frac{1}{12}2^{-k+1}3\log(k+2) \ge \frac{1}{12}|R_j|,$$

which proves the required conclusion (4).

Conclusion (1) in Lemma 5.1.1 follows from the fact that the regions inside the angles $\angle A_jC_jB_j$ and under the triangles $A_jC_jB_j$ are pairwise disjoint. This is shown in Figure 5.6. This can be proved rigorously by a careful examination of the construction of the sprouted triangles $A_jC_jB_j$, but the details are omitted.

It remains to prove (3). To achieve this we first estimate the length of the line segment A_jD_j from below. The law of sines gives

$$\frac{|A_jD_j|}{\sin(\angle A_jB_jD_j)} = \frac{2^{-k}}{\sin(\angle A_jD_jB_j)},$$

from which we obtain that

$$|A_jD_j| \ge \frac{1}{2^k}\sin(\angle A_jB_jD_j) \ge \frac{1}{2^k}\sin(\angle A_0B_0D_0) \ge \frac{1}{2^k}\sin(\angle B_0A_0C_0) = \frac{2^{-k}2}{\sqrt{5}} > \frac{1}{2^{k+1}}.$$

It follows that each R_j has area at least $2^{-k-1}3\log(k+2)$. Therefore,

$$\Big|\bigcup_{j=0}^{2^k-1} R_j\Big| = \sum_{j=0}^{2^k-1}|R_j| \ge 2^k2^{-k-1}3\log(k+2) \ge |E|\log(k+2) \ge \frac{|E|}{\delta},$$

since $|E| \le 3/2$ and k was chosen so that $k+2 > e^{1/\delta}$. □

Next we have a calculation involving the Fourier transforms of characteristic functions of rectangles.

Proposition 5.1.2. *Let R be a rectangle whose center is the origin in $\mathbf{R}^2$ and let v be a unit vector parallel to its longest side. Consider the half-plane*

$$\mathscr{H} = \{x \in \mathbf{R}^2 :\ x\cdot v \ge 0\}$$