$$
\begin{aligned}
|E(\varepsilon,k)| &\le \frac{1}{2}\varepsilon^2 + \sum_{j=1}^{k} 2^j \frac{b_{j-1}}{2} \frac{(h_j - h_{j-1})^2}{2h_j - h_{j-1}} \\
&\le \frac{1}{2}\varepsilon^2 + \sum_{j=1}^{k} 2^j \frac{2^{-(j-1)} b_0}{2} \frac{\varepsilon^2}{(j+1)^2 \varepsilon} \\
&\le \frac{1}{2}\varepsilon^2 + \sum_{j=2}^{\infty} \frac{\varepsilon^2}{j^2} \\
&= \Big(\frac{1}{2} + \frac{\pi^2}{6} - 1\Big)\varepsilon^2 \\
&\le \frac{3}{2}\varepsilon^2 ,
\end{aligned}
$$

where we used the fact that $2h_j - h_{j-1} \ge \varepsilon$ for all $j \ge 1$.

Having completed the construction of the set $E(\varepsilon,k)$, we are now in a position to indicate some of the ideas that appear in the solution of the Kakeya problem. We first observe that no matter what k is, the measure of the set $E(\varepsilon,k)$ can be made as small as we wish if we take ε small enough. Our purpose is to make a needle of infinitesimal width and unit length move continuously from one side of this angle to the other utilizing each sprouted triangle in succession. To achieve this, we need to apply a similar construction to any of the 2^k triangles that make up the set $E(\varepsilon,k)$ and repeat the sprouting procedure a large enough number of times. We refer to [106] for details. An elaborate construction of this sort yields a set within which the needle can be turned only through a fixed angle. But adjoining a few such sets together allows us to rotate a needle through a half-turn within a set that still has arbitrarily small area. This is the idea used to solve the aforementioned needle problem.

5.1.2 The counterexample

We now return to the multiplier problem for the ball, which has an interesting connection with the Kakeya needle problem.

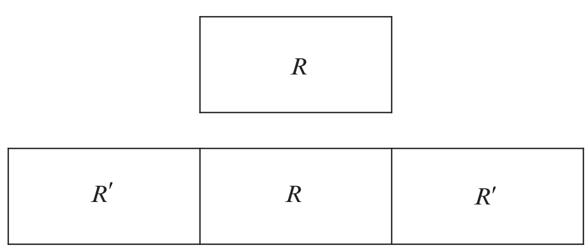

Fig. 5.4 A rectangle R and its adjacent rectangles R'.