Exercises

1.3.1. Let $1 < p < \infty$ and $s \in \mathbf{R}$. Show that the spaces $\dot{L}_s^p$ and L_s^p are complete and that the latter decrease as s increases.

1.3.2. Let $1 < p < \infty$ and $s \in \mathbf{Z}^+$.
(a) Suppose that $f \in L_s^p(\mathbf{R}^n)$ and that φ is in $\mathscr{S}(\mathbf{R}^n)$. Prove that φf is also an element of $L_s^p(\mathbf{R}^n)$.
(b) Let v be a function whose Fourier transform is a bounded compactly supported function. Prove that if f is in $L_s^2(\mathbf{R}^n)$, then so is vf.

1.3.3. Fix $s > 0$ and let α be a multi-index. Let δ_0 be the Dirac mass at the origin on $\mathbf{R}^n$.
(a) If $0 < s - |\alpha| < n$, show that $\partial^\alpha \delta_0 \in L_{-s}^p$ whenever $1 < p < \frac{n}{n+|\alpha|-s}$.
(b) If $n \le s - |\alpha|$, prove that $\partial^\alpha \delta_0 \in L_{-s}^p$ for all $p \in (1, \infty)$.
[*Hint:* Use Proposition 1.2.5.]

1.3.4. Let I be the identity operator, $\mathcal{I}_1$ the Riesz potential of order 1, and R_j the usual Riesz transform. Prove that

$$I = \sum_{j=1}^{n} \mathcal{I}_1 R_j \partial_j \,,$$

and use this identity to obtain Theorem 1.3.5(a) when $s = 1$.
[*Hint:* Take the Fourier transform.]

1.3.5. Let f be in L_s^p for some $1 < p < \infty$. Prove that $\partial^\alpha f$ is in $L_{s-|\alpha|}^p$.

1.3.6. Prove that for all $\mathscr{C}^1$ functions f that are supported in a ball B we have

$$|f(x)| \le \frac{1}{\omega_{n-1}} \int_B |\nabla f(y)| |x-y|^{-n+1} \, dy \,,$$

where $\omega_{n-1} = |\mathbf{S}^{n-1}|$. For such functions obtain the local Sobolev inequality

$$\big\| f \big\|_{L^q(B)} \le C_{q,r,n} \big\| \nabla f \big\|_{L^p(B)} \,,$$

where $1 < p < q < \infty$ and $1/p = 1/q + 1/n$.
[*Hint:* Start from $f(x) = \int_0^\infty \nabla f(x - t\theta) \cdot \theta \, dt$ and integrate over $\theta \in \mathbf{S}^{n-1}$.]

1.3.7. Show that there is a constant C such that for all $\mathscr{C}^1$ functions f that are supported in a ball B we have

$$\frac{1}{|B'|} \int_{B'} |f(x) - f(z)| \, dz \le C \int_B |\nabla f(y)| |x-y|^{-n+1} \, dy$$

for all B' balls contained in B and all $x \in B'$.
[*Hint:* Start with $f(z) - f(x) = \int_0^1 \nabla f(x + t(z-x)) \cdot (z-x) \, dt$.]