are significantly more complicated for two main reasons: the roughness of the variable coefficients of the aforementioned elliptic operator and the higher-dimensional nature of the problem.

4.7.1 Preliminaries and Statement of the Main Result

For $\xi = (\xi_1,\dots,\xi_n) \in \mathbf{C}^n$ we denote its complex conjugate $(\overline{\xi_1},\dots,\overline{\xi_n})$ by $\overline{\xi}$. Moreover, for $\xi,\zeta \in \mathbf{C}^n$ we use the inner product notation

$$\xi\cdot\zeta = \sum_{k=1}^{n} \xi_k\,\zeta_k\,.$$

Throughout this section, $A = A(x)$ is an $n\times n$ matrix of complex-valued L^∞ functions, defined on $\mathbf{R}^n$, that satisfies the *ellipticity* (or *accretivity*) conditions for some $0<\lambda\le\Lambda<\infty$, that

$$\begin{aligned} \lambda|\xi|^2 \;&\le\; \operatorname{Re}\big(A(x)\,\xi\cdot\overline{\xi}\,\big)\,,\\ |A(x)\,\xi\cdot\overline{\zeta}| \;&\le\; \Lambda|\xi|\,|\zeta|\,, \end{aligned} \tag{4.7.1}$$

for all $x\in\mathbf{R}^n$ and $\xi,\zeta\in\mathbf{C}^n$. We interpret an element ξ of $\mathbf{C}^n$ as a column vector in $\mathbf{C}^n$ when the matrix A acts on it.

Associated with such a matrix A, we define a second-order *divergence form operator* on $\mathscr{C}_0^\infty$ functions f by

$$L(f) = -\operatorname{div}(A\nabla f) = -\sum_{j=1}^{n}\partial_j\big((A\nabla f)_j\big)\,. \tag{4.7.2}$$

~~which we interpret in the weak sense whenever f is a distribution.~~

The accretivity condition (4.7.1) enables us to define a square root operator $L^{1/2}=\sqrt{L}$ so that the operator identity $L=\sqrt{L}\sqrt{L}$ holds. The *square root operator* can be written in several ways, one of which is

$$\sqrt{L}(f) = \frac{16}{\pi}\int_0^{+\infty}(I+t^2L)^{-3}t^3L^2(f)\,\frac{dt}{t}\,. \tag{4.7.3}$$

We refer the reader to Exercise 4.7.2 for the existence of the square root operator and the validity of identity (4.7.3).

An important problem in the subject is to determine whether the estimate

$$\big\|\sqrt{L}(f)\big\|_{L^2} \le C_{n,\lambda,\Lambda}\big\|\nabla f\big\|_{L^2} \tag{4.7.4}$$

holds for functions f in $\mathscr{C}_0^\infty$. ~~in a dense subspace of the homogeneous Sobolev space $\dot{L}^2_1(\mathbf{R}^n)$, where $C_{n,\lambda,\Lambda}$ is a constant depending only on n, λ, and Λ. Once (4.7.4) is known for a dense subspace of $\dot{L}^2_1(\mathbf{R}^n)$, then it can be extended to the entire space by density.~~ In proving so, we assume that A is *a priori* smooth, but we obtain estimates that depend only on n, λ, and Λ and not on the smoothness of A. The main purpose of this section is to discuss a detailed proof of the following result.

Theorem 4.7.1. *Let L be as in (4.7.2). Then there is a constant $C_{n,\lambda,\Lambda}$ such that for all smooth functions f with compact support, estimate (4.7.4) is valid, whenever A has smooth coefficients.*

The proof of this theorem requires certain estimates concerning elliptic operators. These are presented in the next subsection, while the proof of the theorem follows in the remaining four subsections.

4.7.2 Estimates for Elliptic Operators on $\mathbf{R}^n$

The following lemma provides a quantitative expression for the mean decay of the resolvent kernel.

Lemma 4.7.2. *Let E and F be two closed sets of $\mathbf{R}^n$. Assume that the distance $d = \mathrm{dist}\,(E,F)$ between them is positive. Then for all complex-valued functions f supported in E and all vector-valued functions $\vec{f}$ supported in E, we have*

$$\int_F |(I+t^2L)^{-1}(f)(x)|^2\,dx \leq Ce^{-c\frac{d}{t}}\int_E |f(x)|^2\,dx, \tag{4.7.5}$$

$$\int_F |t\nabla(I+t^2L)^{-1}(f)(x)|^2\,dx \leq Ce^{-c\frac{d}{t}}\int_E |f(x)|^2\,dx, \tag{4.7.6}$$

$$\int_F |(I+t^2L)^{-1}(t\,\mathrm{div}\,\vec{f}\,)(x)|^2\,dx \leq Ce^{-c\frac{d}{t}}\int_E |\vec{f}(x)|^2\,dx, \tag{4.7.7}$$

where $c = c(\lambda,\Lambda)$, $C = C(n,\lambda,\Lambda)$ are finite constants.

Proof. It suffices to obtain these inequalities whenever $d \geq t > 0$. Let us set $u_t = (I+t^2L)^{-1}(f)$. For all $v \in L^2_1(\mathbf{R}^n)$ we have

$$\int_{\mathbf{R}^n} u_t v\,dx + t^2\int_{\mathbf{R}^n} A\nabla u_t\cdot\nabla v\,dx = \int_{\mathbf{R}^n} fv\,dx.$$

Let η be a nonnegative smooth function with compact support that does not meet E and that satisfies $\|\eta\|_{L^\infty} = 1$. Taking $v = \overline{u_t}\,\eta^2$ and using that f is supported in E, we obtain

$$\int_{\mathbf{R}^n} |u_t|^2\eta^2\,dx + t^2\int_{\mathbf{R}^n} A\nabla u_t\cdot\overline{\nabla u_t}\;\eta^2\,dx = -2t^2\int_{\mathbf{R}^n} A(\eta\nabla u_t)\cdot\overline{u_t\nabla\eta}\,dx.$$

Using (4.7.1) and the inequality $2ab \leq \varepsilon|a|^2+\varepsilon^{-1}|b|^2$, we obtain for all $\varepsilon > 0$,

$$\begin{aligned}\int_{\mathbf{R}^n} |u_t|^2\eta^2\,dx + \lambda t^2\int_{\mathbf{R}^n} |\nabla u_t|^2\;\eta^2\,dx \\ \leq \Lambda\varepsilon t^2\int_{\mathbf{R}^n} |\nabla u_t|^2\;\eta^2\,dx + \Lambda\varepsilon^{-1}t^2\int_{\mathbf{R}^n} |u_t|^2|\nabla\eta|^2\,dx,\end{aligned}$$