writing the operator $\mathcal{C}_\Gamma + I$, acting on the Schwartz class, as an average of smoother operators. Precisely, we have shown that for $h \in \mathscr{S}(\mathbf{R})$ we have

$$\mathcal{C}_\Gamma(h)(x) + h(x) = \int_0^\infty s^2 \frac{d^2}{ds^2}\mathcal{C}_\Gamma(h)(x;s)\,\frac{ds}{s}\,, \tag{4.6.21}$$

and it remains to understand what the operator

$$\frac{d^2}{ds^2}\mathcal{C}_\Gamma(h)(x;s) = \mathcal{C}_\Gamma(h)''(x;s)$$

really is. Differentiating (4.6.16) twice, we obtain

$$\begin{aligned}
\mathcal{C}_\Gamma(h)(x) + h(x) &= \int_0^\infty s^2 \mathcal{C}_\Gamma(h)''(x;s)\,\frac{ds}{s} \\
&= 4\int_0^\infty s^2 \mathcal{C}_\Gamma(h)''(x;2s)\,\frac{ds}{s} \\
&= -\frac{8}{\pi i}\int_0^\infty \int_{\mathbf{R}} \frac{s^2 h(y)(1+iA'(y))}{(y-x+i(A(y)-A(x))+2is)^3}\,dy\,\frac{ds}{s} \\
&= -\frac{8}{\pi i}\int_0^\infty \int_\Gamma \frac{s^2 H(\zeta)}{(\zeta - z + 2is)^3}\,d\zeta\,\frac{ds}{s}\,,
\end{aligned}$$

where in the last step we set $z = x + iA(x)$, $H(z) = h(x)$, and we switched to complex integration over the curve Γ. We now use the following identity from complex analysis. For $\zeta, z \in \Gamma$ we have

$$\frac{1}{(\zeta - z + 2is)^3} = -\frac{1}{4\pi i}\int_\Gamma \frac{1}{(\zeta - w + is)^2}\,\frac{1}{(w - z + is)^2}\,dw\,, \tag{4.6.22}$$

for which we refer to Exercise 4.6.3. Inserting this identity in the preceding expression for $\mathcal{C}_\Gamma(h)(x) + h(x)$, we obtain

$$\mathcal{C}_\Gamma(h)(x) + h(x) = -\frac{2}{\pi^2}\int_0^\infty \left[\int_\Gamma \frac{s}{(w-z+is)^2}\left(\int_\Gamma \frac{s\,H(\zeta)}{(\zeta - w + is)^2}\,d\zeta\right)dw\right]\frac{ds}{s}\,,$$

recalling that $z = x + iA(x)$. Introducing the linear operator

$$\Theta_s(h)(x) = \int_{\mathbf{R}} \theta_s(x,y)\,h(y)\,dy\,, \tag{4.6.23}$$

where

$$\theta_s(x,y) = \frac{s}{(y-x+i(A(y)-A(x))+is)^2}\,, \tag{4.6.24}$$

we may therefore write

$$\mathcal{C}_\Gamma(h)(x) + h(x) = -\frac{2}{\pi^2}\int_0^\infty \Theta_s\big((1+iA')\Theta_s\big((1+iA')h\big)\big)(x)\,\frac{ds}{s}\,. \tag{4.6.25}$$