and that the function $|\xi|^s(1-\widehat{\Phi}(\xi))(1+|\xi|^2)^{-\frac{s}{2}}$ is in $\mathscr{M}_p(\mathbf{R}^n)$ by Theorem 6.2.7 in [156]. It follows that

$$\big\|f_\infty\big\|_{L^p} \le C\big\|f_s\big\|_{L^p} = C\big\|f\big\|_{L^p_s},$$

which, combined with (1.3.19), yields

$$\Big\|\Big(\sum_{j=2}^{\infty}|2^{js}\Delta_j^{\Psi}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \le C\big\|f\big\|_{L^p_s}. \tag{1.3.20}$$

Finally, we have

$$2^s\Delta_1^{\Psi}(f) = 2^s\big(\widehat{\Psi}(\tfrac{1}{2}\xi)(1+|\xi|^2)^{-\frac{s}{2}}\,\widehat{f_s}\,\big)^{\vee},$$

and since the function $\widehat{\Psi}(\frac{1}{2}\xi)(1+|\xi|^2)^{-\frac{s}{2}}$ being smooth with compact support lies in $\mathscr{M}_p(\mathbf{R}^n)$, it follows that

$$\big\|2^s\Delta_1^{\Psi}(f)\big\|_{L^p} \le C\big\|f_s\big\|_{L^p} = C\big\|f\big\|_{L^p_s}. \tag{1.3.21}$$

Combining estimates (1.3.17), (1.3.20), and (1.3.21), we conclude the proof of (1.3.11). □

1.3.3 Littlewood–Paley Characterization of Homogeneous Sobolev Spaces

We now introduce the homogeneous Sobolev spaces $\dot{L}^p_s$. The main difference with the inhomogeneous spaces L^p_s is that elements of $\dot{L}^p_s$ may not themselves be elements of L^p. Another point of differentiation is that elements of homogeneous Sobolev spaces whose differences are polynomials are identified.

For the purposes of the following definition, for $1<p<\infty$ we define $\dot{L}^p(\mathbf{R}^n)$ as the space of all elements in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ such that every equivalence class $\big[$formed from the relationship $u\equiv v$ if $u-v\in\mathscr{P}(\mathbf{R}^n)\big]$ contains a unique representative that belongs to $L^p(\mathbf{R}^n)$. One defines the $\dot{L}^p(\mathbf{R}^n)$ norm of every element of the equivalence class to be the L^p norm of the unique L^p representative. Under this definition we have

$$\big\|f+P\big\|_{\dot{L}^p} = \big\|f\big\|_{\dot{L}^p} = \big\|f\big\|_{L^p}$$

whenever $f\in L^p$ and P is a polynomial.

Definition 1.3.7. Let s be a real number, and let $1<p<\infty$. The *homogeneous Sobolev space* $\dot{L}^p_s(\mathbf{R}^n)$ is defined as the space of all u in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ for which the well-defined distribution

$$(|\xi|^s\widehat{u})^{\vee}$$

coincides with a function in $\dot{L}^p(\mathbf{R}^n)$. For distributions u in $\dot{L}^p_s(\mathbf{R}^n)$ we define

$$\|u\|_{\dot{L}^p_s} = \big\|(|\cdot|^s\widehat{u})^\vee\big\|_{\dot{L}^p(\mathbf{R}^n)}. \tag{1.3.22}$$

As noted earlier, to avoid working with equivalence classes of functions, we identify two distributions in $\dot{L}^p_s(\mathbf{R}^n)$ whose difference is a polynomial. Under this identification, the quantity in (1.3.22) is a norm.

Theorem 1.3.6 also has a homogeneous version.

Theorem 1.3.8. *Let Ψ satisfy (1.3.6), and let Δ_j^Ψ be the Littlewood–Paley operator associated with Ψ. Let $s \in \mathbf{R}$ and $1 < p < \infty$. Then there exists a constant C_1 that depends only on $n,s,p,$ and Ψ such that for all $f \in \dot{L}^p_s(\mathbf{R}^n)$ we have*

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}(2^{js}|\Delta_j^\Psi(f)|)^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \le C_1\big\|f\big\|_{\dot{L}^p_s}. \tag{1.3.23}$$

Conversely, there exists a constant C_2 that depends on the parameters $n,s,p,$ and Ψ such that every element f of $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ that satisfies

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}(2^{js}|\Delta_j^\Psi(f)|)^2\Big)^{\frac{1}{2}}\Big\|_{L^p} < \infty$$

lies in the homogeneous Sobolev space $\dot{L}^p_s$ and we have

$$\big\|f\big\|_{\dot{L}^p_s} \le C_2\Big\|\Big(\sum_{j\in\mathbf{Z}}(2^{js}|\Delta_j^\Psi(f)|)^2\Big)^{\frac{1}{2}}\Big\|_{L^p}. \tag{1.3.24}$$

Proof. The proof of the theorem is similar to but a bit simpler than that of Theorem 1.3.6. To obtain (1.3.23), we start with $f \in \dot{L}^p_s$ and note that

$$2^{js}\Delta_j^\Psi(f) = 2^{js}\big(|\xi|^s|\xi|^{-s}\widehat{\Psi}(2^{-j}\xi)\,\widehat{f}\,\big)^\vee = \big(\widehat{\sigma}(2^{-j}\xi)\,\widehat{f_s}\,\big)^\vee = \Delta_j^\sigma(f_s),$$

where $\widehat{\sigma}(\xi) = \widehat{\Psi}(\xi)|\xi|^{-s}$ and Δ_j^σ is the Littlewood–Paley operator given on the Fourier transform side by multiplication with the function $\widehat{\sigma}(2^{-j}\xi)$. We have

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}|2^{js}\Delta_j^\Psi(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} = \Big\|\Big(\sum_{j\in\mathbf{Z}}|\Delta_j^\sigma(f_s)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \le C\big\|f_s\big\|_{\dot{L}^p} = C\big\|f\big\|_{\dot{L}^p_s},$$

where the last inequality follows from Theorem 6.1.2 in [156]. This proves (1.3.23).

Next we show that if the expression on the right-hand side in (1.3.24) is finite, then the distribution f in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ must lie in the homogeneous Sobolev space $\dot{L}^p_s$ with norm controlled by a multiple of this expression.