One situation in which this operator appears is the following: If $\Gamma$ is a closed simple curve (i.e., a Jordan curve), $\Omega_+$ is the interior-connected component of $\mathbf{C}\setminus\Gamma$, $\Omega_-$ is the exterior-connected component of $\mathbf{C}\setminus\Gamma$, and $f$ is a smooth complex function on $\Gamma$, is it possible to find analytic functions $F_+$ on $\Omega_+$ and $F_-$ on $\Omega_-$, respectively, that have continuous extensions on $\Gamma$ such that their difference is equal to the given $f$ on $\Gamma$? It turns out that a solution of this problem is given by the functions

$$F_+(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - w}\,d\zeta\,, \quad w\in\Omega_+\,,$$

and

$$F_-(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - w}\,d\zeta\,, \quad w\in\Omega_-\,.$$

We ~~are~~ would like to study the case in which the Jordan curve $\Gamma$ passes through infinity, in particular, when it is the graph of a Lipschitz function on $\mathbf{R}$. In this case we compute the boundary limits of $F_+$ and $F_-$ and we see that they give rise to a very interesting operator on the curve $\Gamma$. To fix notation we let

$$A:\ \mathbf{R}\to\mathbf{R}$$

be a Lipschitz function. This means that there is a constant $L>0$ such that for all $x,y\in\mathbf{R}$ we have $|A(x)-A(y)|\le L|x-y|$. We define a curve

$$\gamma:\ \mathbf{R}\to\mathbf{C}$$

by setting

$$\gamma(x) = x + iA(x)$$

and we denote by

$$\Gamma = \{\gamma(x):\ x\in\mathbf{R}\} \tag{4.6.1}$$

the graph of $\gamma$. Given a smooth function $f$ on $\Gamma$ we set

$$F(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - w}\,d\zeta\,, \quad w\in\mathbf{C}\setminus\Gamma\,. \tag{4.6.2}$$

We now show that for $z\in\Gamma$, both $F(z+i\delta)$ and $F(z-i\delta)$ have limits as $\delta\downarrow 0$, and these limits give rise to an operator on the curve $\Gamma$ that we would like to study.

### *4.6.1 Introduction of the Cauchy Integral Operator along a Lipschitz Curve*

Let $f(\zeta)$ be a $\mathscr{C}^1$ function on the curve $\Gamma$ that decays faster than $C|\zeta|^{-1}$ as $|\zeta|\to\infty$. For $z\in\Gamma$ we define the *Cauchy integral of $f$ at $z$* as