4.5.4 Pseudodifferential Operators

We now turn to another elegant application of Lemma 4.5.1 regarding pseudodifferential operators. We first introduce pseudodifferential operators.

Definition 4.5.5. Let $m \in \mathbf{R}$ and $0 \le \rho, \delta \le 1$. A $\mathscr{C}^\infty$ function $\sigma(x,\xi)$ on $\mathbf{R}^n \times \mathbf{R}^n$ is called a *symbol of class* $S^m_{\rho,\delta}$ if for all multi-indices α and β there is a constant $B_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le B_{\alpha,\beta}(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|}. \tag{4.5.24}$$

For $\sigma \in S^m_{\rho,\delta}$, the linear operator

$$T_\sigma(f)(x) = \int_{\mathbf{R}^n} \sigma(x,\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi$$

initially defined for f in $\mathscr{S}(\mathbf{R}^n)$ is called a *pseudodifferential operator* with symbol $\sigma(x,\xi)$.

Example 4.5.6. Let b be a bounded function on $\mathbf{R}^n$. Consider the symbol

$$\sigma_b(x,\xi) = \sum_{j\in\mathbf{Z}} (b * \Psi_{2^{-j}})(x)\widehat{\Psi}(2^{-j}\xi), \tag{4.5.25}$$

where $\widehat{\Psi}$ is a smooth function supported in the annulus $1/2 \le |\xi| \le 2$. It is not hard to see that the symbol σ_b satisfies

$$|\partial_x^\alpha \partial_\xi^\beta \sigma_b(x,\xi)| \le C_{\alpha,\beta}\|b\|_{L^\infty}|\xi|^{-|\beta|+|\alpha|} \tag{4.5.26}$$

for all multi-indices α and β. Indeed, every differentiation in x produces a factor of 2^j, while every differentiation in ξ produces a factor of 2^{-j}. But since $\widehat{\Psi}$ is supported in $\frac{1}{2}\cdot 2^j \le |\xi| \le 2\cdot 2^j$, it follows that $|\xi| \approx 2^j$, which yields (4.5.26). It follows that σ_b is not in any of the classes $S^m_{\rho,\delta}$ introduced in Definition 4.5.5, since σ_b is not necessarily smooth at the origin. However, if we restrict the indices of summation in (4.5.25) to $j \ge 0$, then $|\xi| \approx 1+|\xi|$ and we obtain a symbol of class $S^0_{1,1}$. Note that not all symbols in $S^0_{1,1}$ give rise to bounded operators on L^2. See Exercise 4.5.6.

An example of a symbol in $S^m_{1,0}$ is $(1+|\xi|^2)^{\frac{1}{2}(m+it)}$ when $m,t \in \mathbf{R}$.

We do not plan to embark on a systematic study of pseudodifferential operators here, but we would like to study the L^2 boundedness of symbols of class $S^0_{0,0}$.

Theorem 4.5.7. *Suppose that a symbol σ belongs to the class $S^0_{0,0}$. Then the pseudodifferential operator T_σ with symbol σ, initially defined on $\mathscr{S}(\mathbf{R}^n)$, has a bounded extension on $L^2(\mathbf{R}^n)$.*