We now prove (iii). First we show that for all $x \in H \setminus \{0\}$ the sequence

$$\Big\{ \sum_{j=-N}^{N} T_j(x) \Big\}_N$$

is Cauchy in H. Suppose that this is not the case. This means that there is some $\varepsilon > 0$, $x \in H \setminus \{0\}$, and a subsequence of integers $1 \le N_1 < N_2 < N_3 < \cdots$ such that

$$\big\| \widetilde{T}_k(x) \big\|_H \ge \varepsilon \,, \tag{4.5.9}$$

where we set

$$\widetilde{T}_k(x) = \sum_{N_k < |j| \le N_{k+1}} T_j(x).$$

For any fixed $\omega \in [0,1]$, we apply conclusion (i) to the family of linear operators $\{r_k(\omega)T_j : 1 \le k \le K,\ N_k < |j| \le N_{k+1}\}$, indexed by $\Lambda = \{j \in \mathbf{Z} : N_1 < |j| \le N_{K+1}\}$, which clearly satisfies hypothesis (4.5.1). We obtain

$$\Big\| \sum_{k=1}^{K} r_k(\omega) \sum_{N_k < |j| \le N_{k+1}} T_j(x) \Big\|_H = \Big\| \sum_{k=1}^{K} r_k(\omega) \widetilde{T}_k(x) \Big\|_H \le A \, \|x\|_H \, .$$

Squaring and integrating this inequality with respect to ω in $[0,1]$, and using (4.5.8) with $\widetilde{T}_k$ in the place of T_k and $\{1,2,\dots,K\}$ in the place of Λ, we obtain

$$\sum_{k=1}^{K} \big\| \widetilde{T}_k(x) \big\|_H^2 \le A^2 \|x\|_H^2.$$

But this clearly contradicts (4.5.9) as $K \to \infty$.

We conclude that every sequence

$$\Big\{ \sum_{j=-N}^{N} T_j(x) \Big\}_N$$

is Cauchy in H and thus it converges to $T(x)$ for some linear operator T. In view of conclusion (i), it follows that T is a bounded operator on H with norm at most A. $\square$

Remark 4.5.2. At first sight, it appears strange that the norm of the operator T is independent of the norm of every piece T_j and depends only on the quantity A in (4.5.1). But as observed in the proof, if we take $j = k$ in (4.5.1), we obtain

$$\big\| T_j \big\|_{H \to H}^2 = \big\| T_j T_j^* \big\|_{H \to H} \le \gamma(0) \le A^2 \,;$$

thus the norm of each individual T_j is also controlled by the constant A.

We also note that there wasn't anything special about the role of the index set $\mathbf{Z}$ in Lemma 4.5.1. Indeed, the set $\mathbf{Z}$ can be replaced by any countable group, such as $\mathbf{Z}^k$ for some k. For instance, see Theorem 4.5.7, in which the index set is $\mathbf{Z}^{2n}$.