$$
\begin{aligned}
&= 4\int_{\mathbf{R}^{n+1}_+} \big|F(x,t)\big|^2\, d\mu(x,t) \\
&= 4\int_{\mathbf{R}^{n+1}_+} \big|(\Phi_t * f)(x)\big|^2\, d\mu(x,t) \\
&\le C_n \big\|b\big\|_{BMO}^2 \int_{\mathbf{R}^n} |f(x)|^2\, dx, \qquad (4.4.10)
\end{aligned}
$$

where we used Theorem 3.3.7 in the last inequality.

Since the expression in (4.4.10) is finite, given $\varepsilon > 0$, we can find an $N_0 > 0$ such that

$$
M \ge N \ge N_0 \implies \sum_{N\le |j|\le M} \int_{\mathbf{R}^n} \big|\big(\Delta_j(b)\, S_{j-3}(f)\big)\widehat{\ }(\xi)\big|^2\, d\xi < \varepsilon .
$$

Recalling that

$$
\int_{\mathbf{R}^n} \bigg| \sum_{N\le |j|\le M} \Delta_j(b)(x)\, S_{j-3}(f)(x)\bigg|^2 dx \le 4 \sum_{N\le |j|\le M} \int_{\mathbf{R}^n} \big|\big(\Delta_j(b) S_{j-3}(f)\big)\widehat{\ }(\xi)\big|^2\, d\xi \,,
$$

we conclude that the sequence

$$
\bigg\{ \sum_{|j|\le M} \Delta_j(b) S_{j-3}(f) \bigg\}_M
$$

is Cauchy in $L^2(\mathbf{R}^n)$, and therefore it converges in L^2 to a function $P_b(f)$. The boundedness of P_b on L^2 follows by setting $N = 0$ and letting $M \to \infty$ in (4.4.9). □

4.4.3 Fundamental Properties of Paraproducts

Having established the L^2 boundedness of paraproducts, we turn to their properties. We begin by studying their kernels. The paraproducts P_b are examples of integral operators of the form discussed in Section 4.1. Since P_b is L^2 bounded, it has a distributional kernel W_b. We show that for each b in BMO the distribution W_b coincides with a standard kernel L_b defined on $\mathbf{R}^n \times \mathbf{R}^n \setminus \{(x,x):\ x \in \mathbf{R}^n\}$.

First we study the kernel of the operator $f \mapsto \Delta_j(b) S_{j-3}(f)$ for any $j \in \mathbf{Z}$. We have that

$$
\Delta_j(b)(x) S_{j-3}(f)(x) = \int_{\mathbf{R}^n} L_j(x,y) f(y)\, dy,
$$

where L_j is the integrable function

$$
L_j(x,y) = (b * \Psi_{2^{-j}})(x) 2^{(j-3)n} \Phi(2^{j-3}(x-y)).
$$