Note that $\widehat{\Phi}(\xi)$ is equal to 1 for $|\xi| \le 2 - \frac{2}{7}$, vanishes when $|\xi| \ge 2$, and satisfies

$$\widehat{\Phi}(\xi) + \sum_{j=1}^{\infty} \widehat{\Psi}(2^{-j}\xi) = 1 \tag{1.3.9}$$

for all ξ in $\mathbf{R}^n$. We now introduce an operator S_0^{Φ} by setting

$$S_0^{\Phi}(f) = \Phi * f\,, \tag{1.3.10}$$

for $f \in \mathscr{S}'(\mathbf{R}^n)$. Identity (1.3.9) yields the operator identity

$$S_0^{\Phi} + \sum_{j=1}^{\infty} \Delta_j^{\Psi} = I\,,$$

in which the series converges in $\mathscr{S}'(\mathbf{R}^n)$, in view of Proposition 1.1.6(b).

Having introduced the relevant background, we are now ready to state and prove the following result.

Theorem 1.3.6. *Let Ψ satisfy (1.3.6), Φ be as in (1.3.8), and let Δ_j^{Ψ}, S_0^{Φ} be as in (1.3.7) and (1.3.10), respectively. Fix $s \in \mathbf{R}$ and $1 < p < \infty$. Then there exists a constant C_1 that depends only on n, s, p, Φ, and Ψ such that for all $f \in L_s^p$ we have*

$$\big\|S_0^{\Phi}(f)\big\|_{L^p} + \Big\|\Big(\sum_{j=1}^{\infty}(2^{js}|\Delta_j^{\Psi}(f)|)^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \le C_1\big\|f\big\|_{L_s^p}\,. \tag{1.3.11}$$

Conversely, there exists a constant C_2 that depends on the parameters n, s, p, Φ, and Ψ such that every tempered distribution f that satisfies

$$\big\|S_0^{\Phi}(f)\big\|_{L^p} + \Big\|\Big(\sum_{j=1}^{\infty}(2^{js}|\Delta_j^{\Psi}(f)|)^2\Big)^{\frac{1}{2}}\Big\|_{L^p} < \infty$$

is an element of the Sobolev space L_s^p with norm

$$\big\|f\big\|_{L_s^p} \le C_2\Big(\big\|S_0^{\Phi}(f)\big\|_{L^p} + \Big\|\Big(\sum_{j=1}^{\infty}(2^{js}|\Delta_j^{\Psi}(f)|)^2\Big)^{\frac{1}{2}}\Big\|_{L^p}\Big)\,. \tag{1.3.12}$$

Proof. We denote by C a generic constant that depends on the parameters n, s, p, Φ, and Ψ and that may vary in different occurrences. For a given tempered distribution f we define another tempered distribution f_s by setting

$$f_s = \big((1+|\cdot|^2)^{\frac{s}{2}}\widehat{f}\big)^{\vee}\,,$$

so that we have $\|f\|_{L_s^p} = \|f_s\|_{L^p}$ if $f \in L_s^p$.