where the second equality is justified by the continuity and linearity of T along with the fact that the Riemann sums of the integral in (4.3.29) converge to that integral in $\mathscr{S}$ (a proof of this fact is essentially contained in the proof of Theorem 2.3.21 in [156]). Consequently,

$$\left\langle T(\tau^{x_0}\varphi_R), g\right\rangle = \lim_{k\to\infty}\int_{\mathbf{R}^n} \widehat{\tau^{x_0}\varphi_R}(\xi)\left\langle T\left(\eta_k e^{2\pi i\xi\cdot(\cdot)}\right), g\right\rangle d\xi. \tag{4.3.30}$$

We show that $\left\langle T\left(\eta_k e^{2\pi i\xi\cdot(\cdot)}\right), g\right\rangle$ is uniformly bounded in k for k large. Suppose that g is supported in the ball $\overline{B(0,M)}$. Let $k_0 = 2M$. Then for $k \ge k_0$ write

$$\left\langle T\left(\eta_k e^{2\pi i\xi\cdot(\cdot)}\right), g\right\rangle = \left\langle T\left(e^{2\pi i\xi\cdot(\cdot)}\right), g\right\rangle - \left\langle T\left((1-\eta_k)e^{2\pi i\xi\cdot(\cdot)}\right), g\right\rangle \tag{4.3.31}$$

The first expression on the right in (4.3.31) is bounded by $B_5\|g\|_{H^1}$, while the second expression can be written as

$$\int_{|y|\ge k}\left[\int_{\mathbf{R}^n}\big(K(x,y)-K(0,y)\big)g(x)\,dx\right]\big(1-\eta_k(y)\big)e^{2\pi i\xi\cdot y}dy,$$

in view of Definition 4.1.16. As $|x| \le \frac{1}{2}\max(|x-y|,|y|)$ when $|y|\ge k\ge k_0\ge 2M$ and $|x|\le M$, we use (4.1.2) to bound the absolute value of the preceding expression by

$$2\|g\|_{L^\infty}\int_{|y|\ge 2M}\int_{|x|\le M}\frac{A|x|^\delta}{|x-y|^{n+\delta}}\,dx\,dy = C' < \infty.$$

The Lebesgue dominated convergence theorem allows us to pass the limit inside the integrals in (4.3.30) to obtain

$$\left\langle T(\tau^{x_0}\varphi_R), g\right\rangle = \int_{\mathbf{R}^n} \widehat{\tau^{x_0}\varphi_R}(\xi)\left\langle T\left(e^{2\pi i\xi\cdot(\cdot)}\right), g\right\rangle d\xi.$$

We now use assumption (v). The distributions $T\left(e^{2\pi i\xi\cdot(\cdot)}\right)$ coincide with BMO functions whose norm is at most B_5. It follows that

$$\begin{aligned}\left|\left\langle T(\tau^{x_0}\varphi_R), g\right\rangle\right| &\le \left\|\widehat{\tau^{x_0}\varphi_R}\right\|_{L^1}\sup_{\xi\in\mathbf{R}^n}\left\|T\left(e^{2\pi i\xi\cdot(\cdot)}\right)\right\|_{BMO}\|g\|_{H^1}\\ &\le C_n B_5 R^{-n}\|g\|_{H^1},\end{aligned} \tag{4.3.32}$$

where the constant C_n is independent of the normalized bump φ in view of (4.3.1). It follows from (4.3.32) that

$$g \mapsto \left\langle T(\tau^{x_0}\varphi_R), g\right\rangle$$

is a bounded linear functional on BMO with norm at most a multiple of B_5R^{-n}. It follows from Theorem 3.2.2 that $T(\tau^{x_0}\varphi_R)$ coincides with a BMO function that satisfies

$$R^n\left\|T(\tau^{x_0}\varphi_R)\right\|_{BMO} \le C_n B_5.$$

The same argument is valid for T^t, and this shows that

$$B_6 \le C_{n,\delta}(A+B_5)$$