where x_0 lies in the support of φ. In the outer integral above we have $y \notin \operatorname{supp} \varphi$ and the inner integral above is absolutely convergent and equal to

$$\int_{\mathbf{R}^n} \big(K(x,y) - K(x_0,y)\big)\varphi(x)\,dx = \int_{\mathbf{R}^n} K^t(y,x)\varphi(x)\,dx = T^t(\varphi)(y)\,,$$

by Proposition 4.1.9, since $y \notin \operatorname{supp} \varphi$. Thus (4.2.12) is valid. □

Exercises

4.2.1. Let $T : \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}'(\mathbf{R}^n)$ be a continuous linear operator whose Schwartz kernel coincides with a function $K(x,y)$ on $\mathbf{R}^n \times \mathbf{R}^n$ minus its diagonal. Suppose that the function $K(x,y)$ satisfies

$$\sup_{R>0} \int_{R \le |x-y| \le 2R} |K(x,y)|\,dy = A < \infty\,.$$

(a) Show that the previous condition is equivalent to

$$\sup_{R>0} \frac{1}{R} \int_{|x-y| \le R} |x-y|\,|K(x,y)|\,dy = A' < \infty$$

by proving that $\frac{1}{2}A' \le A \le 2A'$.
(b) For $\varepsilon > 0$, let $T^{(\varepsilon)}$ be the truncated linear operators with kernels $K^{(\varepsilon)}(x,y) = K(x,y)\chi_{|x-y|>\varepsilon}$. Show that the integral defining $T^{(\varepsilon)}(f)$ converges absolutely for Schwartz functions f.
[*Hint:* Part (b): Consider the annuli $\varepsilon 2^j \le |x| \le \varepsilon 2^{j+1}$ for $j \ge 0$.]

4.2.2. Let T be as in Exercise 4.2.1. Prove that the limit $T^{(\varepsilon)}(f)(x)$ exists for all f in the Schwartz class for almost all $x \in \mathbf{R}^n$ as $\varepsilon \to 0$ if and only if the limit

$$\lim_{\varepsilon \to 0} \int_{\varepsilon < |x-y| < 1} K(x,y)\,dy$$

exists for almost all $x \in \mathbf{R}^n$.

4.2.3. Let $K(x,y)$ be a function defined away from the diagonal in $\mathbf{R}^{2n}$ that satisfies

$$\sup_{R>0} \int_{R \le |x-y| \le 2R} |K(x,y)|\,dx \le A < \infty$$

and also *Hörmander's condition*

$$\sup_{\substack{y,y' \in \mathbf{R}^n \\ y \ne y'}} \int_{|x-y| \ge 2|y-y'|} |K(x,y) - K(x,y')|\,dx \le A'' < \infty\,. \tag{4.2.13}$$