it follows that

$$\left\|(\widehat{f}(\xi)(1+|\xi|^2)^{\frac{k}{2}})^\vee\right\|_{L^p} \le C_{p,n,k} \sum_{|\gamma|\le k} \left\|(\widehat{f}(\xi)\xi^\gamma)^\vee\right\|_{L^p} < \infty .$$

Example 1.3.3. Every Schwartz function lies in $L^p_s(\mathbf{R}^n)$ for s real. Sobolev spaces with negative indices can indeed contain tempered distributions that are not locally integrable functions. For example, consider the Dirac mass at the origin δ_0. Then $\|\delta_0\|_{L^p_{-s}(\mathbf{R}^n)} = (2\pi)^{\frac{n}{p}-n}\|G_s\|_{L^p}$ when $s>0$. For $s \ge n$ this quantity is always finite in view of Proposition 1.2.5. For $0<s<n$ the function $G_s(x) = \big((1+|\xi|^2)^{-\frac{s}{2}}\big)^\vee(x)$ is integrable to the power p as long as $(s-n)p > -n$, that is, when $1<p<\frac{n}{n-s}$. We conclude that δ_0 lies in $L^p_{-s}(\mathbf{R}^n)$ for $1<p<\frac{n}{n-s}$ when $0<s<n$ and in $L^p_{-s}(\mathbf{R}^n)$ for all $1<p<\infty$ when $s \ge n$.

Example 1.3.4. We consider the function $h(t) = 1-t$ for $0 \le t \le 1$, $h(t) = 1+t$ for $-1 \le t < 0$, and $h(t) = 0$ for $|t|>1$. Obviously, the distributional derivative of h is the function $h'(t) = \chi_{(-1,0)} - \chi_{(0,1)}$. The distributional second derivative h'' is equal to $\delta_1 + \delta_{-1} - 2\delta_0$; see Exercise 2.3.4(a) in [156]. Clearly, h'' does not belong to any L^p space; hence h is not in $L^p_2(\mathbf{R})$. But for $1<p<\infty$, h lies in $L^p_1(\mathbf{R})$, and we thus have an example of a function in $L^p_1(\mathbf{R})$ but not in $L^p_2(\mathbf{R})$.

Definition 1.3.2 allows us to fine-tune the smoothness of h by finding all s for which h lies in $L^p_s(\mathbf{R})$. An easy calculation gives

$$\widehat{h}(\xi) = \frac{e^{2\pi i\xi} + e^{-2\pi i\xi} - 2}{4\pi^2|\xi|^2} .$$

Fix a smooth function φ that is equal to one in a neighborhood of infinity and vanishes in the interval $[-2,2]$. Then $\big(\widehat{h}(\xi)(1+|\xi|^2)^{s/2}(1-\varphi(\xi))\big)^\vee$ is the inverse Fourier transform of a smooth function with compact support; hence it is a Schwartz function and belongs to all L^p spaces. It suffices to examine for which p the function

$$u = \left((1+|\xi|^2)^{s/2}\,\frac{e^{2\pi i\xi} + e^{-2\pi i\xi} - 2}{4\pi^2(1+|\xi|^2)}\,\varphi(\xi)\,\frac{1+|\xi|^2}{|\xi|^2}\right)^\vee \tag{1.3.4}$$

lies in $L^p(\mathbf{R})$. We first observe that the function u in (1.3.4) lies in $L^p(\mathbf{R})$ if and only if the function

$$v = \left((1+|\xi|^2)^{s/2}\,\frac{e^{2\pi i\xi} + e^{-2\pi i\xi} - 2}{4\pi^2(1+|\xi|^2)}\right)^\vee \tag{1.3.5}$$

lies in $L^p(\mathbf{R})$. Indeed, if v lies in L^p, then u lies in L^p for $1<p<\infty$ in view of Theorem 6.2.7 in [156], since the bounded function $m(\xi) = \varphi(\xi)\frac{1+|\xi|^2}{|\xi|^2}$ satisfies the Mihlin condition $|m'(\xi)| \le C|\xi|^{-1}$. Conversely, if u lies in L^p, then

$$v = \big(\widehat{v}(\xi)\,(1-\varphi)(\xi)\big)^\vee + \left(\widehat{u}(\xi)\,\frac{|\xi|^2}{1+|\xi|^2}\right)^\vee ,$$