Proposition 4.2.3. *Let T be an operator in $CZO(\delta,A,B)$ associated with a kernel K. Then for $g \in L^p(\mathbf{R}^n)$, $1 \le p < \infty$, the following absolutely convergent integral representation is valid:*

$$T(g)(x) = \int_{\mathbf{R}^n} K(x,y)\, g(y)\, dy \tag{4.2.1}$$

for almost all $x \in \mathbf{R}^n \setminus$ supp g, provided that supp $g \subsetneq \mathbf{R}^n$.

Proof. Set $g_k(x) = g(x)\chi_{|g(x)|\le k}\chi_{|x|\le k}$. These are L^p functions with compact support contained in the support of g. Also, the g_k converge to g in L^p as $k \to \infty$. In view of Proposition 4.1.9, for every k we have

$$T(g_k)(x) = \int_{\mathbf{R}^n} K(x,y)\, g_k(y)\, dy$$

for almost all $x \in \mathbf{R}^n \setminus$ supp g. Since T maps L^p to L^p (or to weak L^1 when $p = 1$), it follows that $T(g_k)$ converges to $T(g)$ in weak L^p and hence in measure. By Proposition 1.1.9 in [156], a subsequence of $T(g_k)$ converges to $T(g)$ almost everywhere. On the other hand, for $x \in \mathbf{R}^n \setminus$ supp g we have

$$\int_{\mathbf{R}^n} K(x,y)\, g_k(y)\, dy \to \int_{\mathbf{R}^n} K(x,y)\, g(y)\, dy$$

when $k \to \infty$, since the absolute value of the difference is bounded by $B'\|g_k - g\|_{L^p}$, which tends to zero. The constant B' is the $L^{p'}$ norm of the function $|x-y|^{-n}$ on the support of g; one has $|x-y| \ge c > 0$ for all y in the support of g and thus $B' < \infty$. Therefore $T(g_k)(x)$ converges a.e. to both sides of the identity (4.2.1) for x not in the support of g. This concludes the proof of this identity. □

4.2.2 Boundedness of Maximal Singular Integrals

We pose the question whether there is a result concerning the maximal singular integral operator $T^{(*)}$ analogous to Theorem 4.2.2. We note that given f in $L^p(\mathbf{R}^n)$ for some $1 \le p < \infty$, the expression $T^{(*)}(f)(x)$ is well defined for all $x \in \mathbf{R}^n$. This is a simple consequence of estimate (4.1.1) and Hölder's inequality.

Theorem 4.2.4. *Let K be in $SK(\delta,A)$ and T in $CZO(\delta,A,B)$ be associated with K. Let $r \in (0,1)$. Then there is a constant $C(n,r)$ such that* Cotlar's inequality

$$|T^{(*)}(f)(x)| \le C(n,r)\Big[M(|T(f)|^r)(x)^{\frac{1}{r}} + (A+B)M(f)(x)\Big] \tag{4.2.2}$$

is valid for all functions in $\bigcup_{1\le p<\infty} L^p(\mathbf{R}^n)$ *and all* $x \in \mathbf{R}^n$. *Also, there exist dimensional constants* C_n, C_n' *such that*

$$\big\|T^{(*)}(f)\big\|_{L^{1,\infty}(\mathbf{R}^n)} \le C_n'(A+B)\big\|f\big\|_{L^1(\mathbf{R}^n)}, \tag{4.2.3}$$

$$\big\|T^{(*)}(f)\big\|_{L^p(\mathbf{R}^n)} \le C_n(A+B)\max(p,(p-1)^{-1})\big\|f\big\|_{L^p(\mathbf{R}^n)}, \tag{4.2.4}$$

for all $1<p<\infty$ *and all* f *in* $L^p(\mathbf{R}^n)$.

Proof. We fix r so that $0<r<1$ and $f \in L^p(\mathbf{R}^n)$ for some p satisfying $1\le p<\infty$. To prove (4.2.2), we fix $x \in \mathbf{R}^n$, $\varepsilon>0$, and we set $f_0^{\varepsilon,x} = f\chi_{B(x,\varepsilon)}$ and $f_\infty^{\varepsilon,x} = f\chi_{B(x,\varepsilon)^c}$. Since $x \notin \operatorname{supp} f_\infty^{\varepsilon,x}$, using Proposition 4.2.3 we should be able to write

$$T(f_\infty^{\varepsilon,x})(x) = \int_{\mathbf{R}^n} K(x,y)\, f_\infty^{\varepsilon,x}(y)\,dy = \int_{|x-y|\ge\varepsilon} K(x,y)\,f(y)\,dy = T^{(\varepsilon)}(f)(x).$$

But as identity (4.2.1) holds a.e., this may not be valid for our fixed $x \in \mathbf{R}^n$. To address this issue, we pick $w_j \in B(x,\varepsilon)$ such that $w_j \to x$ and

$$T(f_\infty^{\varepsilon,x})(w_j) = \int_{\mathbf{R}^n} K(w_j,y) f_\infty^{\varepsilon,x}(y)\,dy = \int_{|x-y|\ge\varepsilon} K(w_j,y)f(y)\,dy.$$

If y is such $|x-y|\ge\varepsilon$, then for j large we have $|z-w_j|\le\frac12|w_j-y|$ when $z\in B(x,\frac{\varepsilon}{2})$. In view of (4.2.1), Fatou's lemma, and (4.1.2), for almost all $z\in B(x,\frac{\varepsilon}{2})$ we have

$$\begin{aligned}
\limsup_{j\to\infty}|T(f_\infty^{\varepsilon,x})(w_j)-T(f_\infty^{\varepsilon,x})(z)| &= \limsup_{j\to\infty}\bigg|\int_{|x-y|\ge\varepsilon}\big(K(w_j,y)-K(z,y)\big)f(y)\,dy\bigg| \\
&\le \int_{|x-y|\ge\varepsilon}\limsup_{j\to\infty}\frac{A\,|z-w_j|^\delta\,|f(y)|}{(|w_j-y|+|z-y|)^{n+\delta}}\,dy \\
&\le \Big(\frac{\varepsilon}{2}\Big)^\delta\int_{|x-y|\ge\varepsilon}\frac{A\,|f(y)|\,dy}{(|x-y|+\varepsilon/2)^{n+\delta}} \\
&\le C_{n,\delta}\,A\,M(f)(x),
\end{aligned}$$

where in the last estimate we made use of Theorem 2.1.10 in [156]. Thus

$$|T^{(\varepsilon)}(f)(x)| \le \limsup_{j\to\infty}|T(f_\infty^{\varepsilon,x})(w_j)-T(f_\infty^{\varepsilon,x})(z)|+|T(f_\infty^{\varepsilon,x})(z)|,$$

and from this we derive that for almost all $z\in B(x,\frac{\varepsilon}{2})$ one has

$$|T^{(\varepsilon)}(f)(x)| \le C_{n,\delta}\,A\,M(f)(x)+|T(f_0^{\varepsilon,x})(z)|+|T(f)(z)|. \tag{4.2.5}$$

For $0<r<1$ it follows from (4.2.5) that for almost all $z\in B(x,\frac{\varepsilon}{2})$ we have

$$|T^{(\varepsilon)}(f)(x)|^r \le C_{n,\delta}^r A^r M(f)(x)^r + |T(f_0^{\varepsilon,x})(z)|^r + |T(f)(z)|^r. \tag{4.2.6}$$

Integrating over $z\in B(x,\frac{\varepsilon}{2})$, dividing by $|B(x,\frac{\varepsilon}{2})|$, and raising to the power $\frac1r$, we obtain

$$\begin{aligned}
|T^{(\varepsilon)}(f)(x)| \le 3^{\frac1r}\Bigg[C_{n,\delta}\,A\,M(f)(x)+\bigg(\frac{1}{|B(x,\frac{\varepsilon}{2})|}\int_{B(x,\frac{\varepsilon}{2})}|T(f_0^{\varepsilon,x})(z)|^r dz\bigg)^{\frac1r} \\
+M(|T(f)|^r)(x)^{\frac1r}\Bigg].
\end{aligned}$$