where the limit is taken in the weak topology of L^2, so that T is equal to T_0 plus a bounded function times the identity operator.

We give a special name to operators of this form.

Definition 4.1.13. Suppose that for a given T in $CZO(\delta,A,B)$ there is a sequence ε_j of positive numbers that tends to zero as $j\to\infty$ such that for all $f\in L^2(\mathbf{R}^n)$,

$$T^{(\varepsilon_j)}(f)\to T(f)$$

weakly in L^2. Then T is called a *Calderón–Zygmund singular integral operator*. Thus Calderón–Zygmund singular integral operators are special kinds of Calderón–Zygmund operators. The subclass of $CZO(\delta,A,B)$ consisting of all Calderón–Zygmund singular integral operators is denoted by $CZSIO(\delta,A,B)$.

In view of Proposition 4.1.11 and Remark 4.1.12, a Calderón–Zygmund operator is equal to a Calderón–Zygmund singular integral operator plus a bounded function times the identity operator. For this reason, the study of Calderón–Zygmund operators is equivalent to the study of Calderón–Zygmund singular integral operators, and in almost all situations it suffices to restrict attention to the latter.

4.1.3 Calderón–Zygmund Operators Acting on Bounded Functions

We are now interested in defining the action of a Calderón–Zygmund operator T on bounded and smooth functions. To achieve this we first need to define the space of special test functions $\mathscr{D}_0$.

Definition 4.1.14. We denote by $\mathscr{D}(\mathbf{R}^n)=\mathscr{C}_0^\infty(\mathbf{R}^n)$ the space of all smooth functions with compact support on $\mathbf{R}^n$. We define $\mathscr{D}_0(\mathbf{R}^n)$ to be the space of all smooth functions with compact support and integral zero. We equip $\mathscr{D}_0(\mathbf{R}^n)$ with the same topology as the space $\mathscr{D}(\mathbf{R}^n)$. This means that a linear functional $u\in\mathscr{D}_0'(\mathbf{R}^n)$ is continuous if for any compact set K in $\mathbf{R}^n$ there is a constant C_K and an integer M such that

$$\big|\langle u,\varphi\rangle\big|\le C_K\sum_{|\alpha|\le M}\|\partial^\alpha\varphi\|_{L^\infty}$$

for all φ smooth functions supported in K. The dual space of $\mathscr{D}_0(\mathbf{R}^n)$ under this topology is denoted by $\mathscr{D}_0'(\mathbf{R}^n)$. This is a space of distributions larger than $\mathscr{D}'(\mathbf{R}^n)$.

Example 4.1.15. *BMO* functions are examples of elements of $\mathscr{D}_0'(\mathbf{R}^n)$. Indeed, given $b\in BMO(\mathbf{R}^n)$, for any compact set K there is a constant $C_K=\|b\|_{L^1(K)}$ such that

$$\left|\int_{\mathbf{R}^n}b(x)\varphi(x)\,dx\right|\le C_K\big\|\varphi\big\|_{L^\infty}$$

for any $\varphi\in\mathscr{D}_0(\mathbf{R}^n)$. Moreover, observe that the preceding integral remains unchanged if the *BMO* function b is replaced by $b+c$, where c is a constant.