for some function $T_0^{f_k}$ in $L^2(\mathbf{R}^n)$. Moreover, each $\{\varepsilon_j^k\}_{j=1}^\infty$ can be chosen to be a subsequence of $\{\varepsilon_j^{k-1}\}_{j=1}^\infty$, $k \geq 2$. Then the diagonal sequence $\{\varepsilon_j^j\}_{j=1}^\infty = \{\varepsilon_j\}_{j=1}^\infty$ satisfies

$$\lim_{j\to\infty} \int_{\mathbf{R}^n} T^{(\varepsilon_j)}(f_k)(x)g(x)\,dx = \int_{\mathbf{R}^n} T_0^{f_k}(x)g(x)\,dx \tag{4.1.24}$$

for each k and $g \in L^2$. Since $\{f_k\}_{k=1}^\infty$ is dense in $L^2(\mathbf{R}^n)$, a standard $\varepsilon/3$ argument gives that the sequence of complex numbers

$$\int_{\mathbf{R}^n} T^{(\varepsilon_j)}(f)(x)g(x)\,dx$$

is Cauchy and thus it converges. Now L^2 is complete[1] in the weak* topology; therefore for each $f \in L^2(\mathbf{R}^n)$ there is a function $T_0(f)$ such that (4.1.22) holds for all f, g in $L^2(\mathbf{R}^n)$ as $j \to \infty$. It is easy to see that T_0 is a linear operator with the property $T_0(f_k) = T_0^{f_k}$ for each $k = 1, 2, \dots$. This proves (2).

The L^2 boundedness of T_0 is a consequence of (4.1.22), (4.1.21), and duality, since

$$\big\|T_0(f)\big\|_{L^2} \leq \sup_{\|g\|_{L^2}\leq 1} \limsup_{j\to\infty} \left| \int_{\mathbf{R}^n} T^{(\varepsilon_j)}(f)(x)g(x)\,dx \right| \leq B' \big\|f\big\|_{L^2}.$$

This proves (3). Finally, (1) is a consequence of the integral representation

$$\int_{\mathbf{R}^n} T^{(\varepsilon_j)}(f)(x)g(x)\,dx = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} K^{(\varepsilon_j)}(x,y)f(y)\,dy\,g(x)\,dx,$$

whenever f, g are Schwartz functions with disjoint supports, by letting $j \to \infty$.

We finally prove (4). We first observe that if g is a bounded function with compact support and Q is an open cube in $\mathbf{R}^n$, we have

$$(T^{(\varepsilon_j)} - T)(g\chi_Q)(x) = \chi_Q(x)\,(T^{(\varepsilon_j)} - T)(g)(x)\,, \tag{4.1.25}$$

for almost all $x \notin \partial Q$ whenever ε_j is small enough (depending on x). Indeed, since $g\chi_Q$ is bounded and has compact support, by the integral representation formula (4.1.18) in Proposition 4.1.9 there is a null set $E(g\chi_Q)$ such that for $x \notin \overline{Q} \cap E(g\chi_Q)$ and for $\varepsilon_j < \operatorname{dist}(x, \operatorname{supp} g\chi_Q)$, the left-hand side in (4.1.25) is zero, since in this case x is not in the support of $g\chi_Q$. Moreover, since $g\chi_{Q^c}$ is also bounded and compactly supported, there is a null set $E(g\chi_{Q^c})$ such that for $x \in Q \cap E(g\chi_{Q^c})$ and $\varepsilon_j < \operatorname{dist}(x, \operatorname{supp} g\chi_{Q^c})$ we have that x does not lie in the support of $g\chi_{Q^c}$, and thus $(T^{(\varepsilon_j)} - T)(g\chi_{Q^c})(x) = 0$; hence (4.1.25) holds in this case as well. This proves (4.1.25) for almost all x not in the boundary ∂Q. Taking weak limits in (4.1.25) as $\varepsilon_j \to 0$, we obtain that

$$(T_0 - T)(g\chi_Q) = \chi_Q\,(T_0 - T)(g) \qquad \text{a.e.} \tag{4.1.26}$$

[1] the unit ball of L^2 in the weak* topology is compact and metrizable, hence complete.