Indeed, define

$$\big\langle W,F\big\rangle=\lim_{\varepsilon\to 0}\iint\limits_{|x-y|>\varepsilon}K(x,y)F(x,y)\,dy\,dx \tag{4.1.10}$$

for all F in the Schwartz class of $\mathbf{R}^{2n}$. In view of antisymmetry, we may write

$$\iint\limits_{|x-y|>\varepsilon}K(x,y)F(x,y)\,dy\,dx=\frac{1}{2}\iint\limits_{|x-y|>\varepsilon}K(x,y)\big(F(x,y)-F(y,x)\big)\,dy\,dx\,.$$

In view of (4.1.1), the observation that

$$|F(x,y)-F(y,x)|\le\frac{2\,|x-y|}{(1+|x|^2+|y|^2)^{n+1}}\sup_{(t,s)\in\mathbf{R}^{2n}}\Big|\nabla_{t,s}\Big((1+|t|^2+|s|^2)^{n+1}F(t,s)\Big)\Big|,$$

and the fact that the preceding supremum is controlled by a finite sum of Schwartz seminorms of F, the limit in (4.1.10) exists and gives a tempered distribution on $\mathbf{R}^{2n}$. We can therefore define an operator $T:\mathscr{S}(\mathbf{R}^n)\to\mathscr{S}'(\mathbf{R}^n)$ with kernel W via

$$\begin{aligned}\big\langle T(f),\varphi\big\rangle&=\lim_{\varepsilon\to 0}\iint\limits_{|x-y|>\varepsilon}K(x,y)f(y)\varphi(x)\,dy\,dx\\&=\frac{1}{2}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}K(x,y)[f(y)\varphi(x)-f(x)\varphi(y)]\,dy\,dx\,,\end{aligned} \tag{4.1.11}$$

for all $f,\varphi\in\mathscr{S}(\mathbf{R}^n)$.

Example 4.1.6. Let A be a real-valued Lipschitz function on $\mathbf{R}$. This means that it satisfies the estimate $|A(x)-A(y)|\le L|x-y|$ for some $L<\infty$ and all $x,y\in\mathbf{R}$. For $x,y\in\mathbf{R}$, $x\neq y$, we let

$$K_A(x,y)=\frac{1}{x-y+i(A(x)-A(y))}\,. \tag{4.1.12}$$

A simple calculation gives that when $|y-y'|\le\frac{1}{2}\max\big(|x-y|,|x-y'|\big)$ then

$$|K_A(x,y)-K_A(x,y')|\le\frac{|y-y'|+|A(y)-A(y')|}{|x-y||x-y'|}\le\frac{(1+L)|y-y'|}{\frac{1}{8}(|x-y|+|x-y'|)^2}$$

where the last inequality uses the observation in Remark 4.1.1. Since K_A is antisymmetric, it follows that it is a standard kernel in $SK(1,8(1+L))$.

Example 4.1.7. Let the function A be as in the previous example. For each integer $m\ge 1$ and $x,y\in\mathbf{R}$ we set

$$K_m(x,y)=\left(\frac{A(x)-A(y)}{x-y}\right)^m\frac{1}{x-y}\,. \tag{4.1.13}$$

Clearly, K_m is an antisymmetric function. To see that each K_m is a standard kernel, notice that when $|y-y'| \le \frac{1}{2}\max\big(|x-y|,|x-y'|\big)$ we have

$$\begin{aligned}\left|\frac{A(x)-A(y)}{x-y}-\frac{A(x)-A(y')}{x-y'}\right| &= \left|\frac{(x-y)(A(y')-A(y))+(y-y')(A(x)-A(y))}{(x-y)(x-y')}\right| \\ &\le 2L\,\frac{|y-y'|}{|x-y'|}\,.\end{aligned}$$

Combining this fact with $|a^m-b^m| \le |a-b|(|a|^{m-1}+|a|^{m-2}|b|+\cdots+|b|^{m-1})$ we obtain

$$\begin{aligned}&\big|K_m(x,y)-K_m(x,y')\big| \\ &\le \left|\left(\frac{A(x)-A(y)}{x-y}\right)^m-\left(\frac{A(x)-A(y')}{x-y'}\right)^m\right|\frac{1}{|x-y|}+\left|\frac{A(x)-A(y')}{x-y'}\right|^m\left|\frac{1}{x-y}-\frac{1}{x-y'}\right| \\ &\le \frac{2L|y-y'|}{|x-y'|}mL^{m-1}\frac{1}{|x-y|}+L^m\frac{|y-y'|}{|x-y|\,|x-y'|} \\ &= \frac{(2m+1)L^m|y-y'|}{|x-y|\,|x-y'|} \\ &\le \frac{8(2m+1)L^m|y-y'|}{(|x-y|+|x-y'|)^2}\,.\end{aligned}$$

It follows that K_m lies in $SK(\delta,C)$ with $\delta=1$ and $C=8(2m+1)L^m$. The linear operator with kernel $(\pi i)^{-1}K_m$ is called the mth *Calderón commutator*.

4.1.2 Operators Associated with Standard Kernels

Having introduced standard kernels, we are in a position to define linear operators associated with them.

Definition 4.1.8. Let $0<\delta,A<\infty$, and K in $SK(\delta,A)$. A continuous linear operator T from $\mathscr{S}(\mathbf{R}^n)$ to $\mathscr{S}'(\mathbf{R}^n)$ is said to be *associated with* K if it satisfies

$$T(f)(x)=\int_{\mathbf{R}^n}K(x,y)f(y)\,dy \tag{4.1.14}$$

for all $f\in\mathscr{C}_0^\infty$ and x not in the support of f. If T is associated with K, then the Schwartz kernel W of T coincides with K on $\mathbf{R}^n\times\mathbf{R}^n\setminus\{(x,x):\ x\in\mathbf{R}^n\}$.

If T is associated with K and satisfies

$$\big\|T(\varphi)\big\|_{L^2}\le B\,\|\varphi\|_{L^2} \tag{4.1.15}$$

for all $\varphi\in\mathscr{S}(\mathbf{R}^n)$, then T is called a *Calderón–Zygmund operator* associated with the standard kernel K. Such operators T admit a bounded extension on $L^2(\mathbf{R}^n)$, i.e., given any f in $L^2(\mathbf{R}^n)$ one can define $T(f)$ as the unique L^2 limit of the Cauchy sequence $\{T(\varphi_k)\}_k$, where $\varphi_k \in \mathscr{S}(\mathbf{R}^n)$ and φ_k converges to f in L^2. In this case we keep the same notation for the L^2 extension of T.

In the sequel we denote by $CZO(\delta,A,B)$ the class of all Calderón–Zygmund operators associated with standard kernels in $SK(\delta,A)$ that admit L^2–bounded extensions with norm at most B.

We make the point that there may be several Calderón–Zygmund operators associated with a given standard kernel K. For instance, we may check that the zero operator and the identity operator have the same kernel $K(x,y)=0$. We investigate connections between any two such operators in Proposition 4.1.11. Next we discuss the important fact that once an operator T admits an extension that is L^2 bounded, then (4.1.14) holds for all f that are bounded and compactly supported whenever the point x does not lie in its support.

Proposition 4.1.9. *Let T be an element of $CZO(\delta,A,B)$ associated with a standard kernel K. Then for every f and φ bounded and compactly supported functions that satisfy*

$$\operatorname{dist}(\operatorname{supp}\varphi,\operatorname{supp} f)>0, \tag{4.1.16}$$

then we have the (absolutely convergent) integral representation

$$\int_{\mathbf{R}^n} T(f)(x)\,\varphi(x)\,dx = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} K(x,y)f(y)\varphi(x)\,dy\,dx. \tag{4.1.17}$$

Moreover, given any bounded function with compact support f, there is a set of measure zero $E(f)$ such that $x_0 \notin E(f)\cup \operatorname{supp} f$ we have the (absolutely convergent) integral representation

$$T(f)(x_0) = \int_{\mathbf{R}^n} K(x_0,y)f(y)\,dy. \tag{4.1.18}$$

Proof. We first prove (4.1.17). Given f and φ bounded functions with compact support select $f_j,\varphi_j \in \mathscr{C}_0^\infty$ such that φ_j are uniformly bounded and supported in a small neighborhood of the support of φ, $\varphi_j \to \varphi$ in L^2 and almost everywhere, $f_j \to f$ in L^2 and almost everywhere, and

$$\operatorname{dist}(\operatorname{supp}\varphi_j,\operatorname{supp} f_j) \ge \frac{1}{2}\operatorname{dist}(\operatorname{supp}\varphi,\operatorname{supp} f)=c>0$$

for all $j\in\mathbf{Z}^+$. In view of (4.1.7), identity (4.1.17) is valid for the functions f_j and φ_j in place of f and φ, i.e.,

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} K(x,y)f_j(y)\varphi_j(x)\,dy\,dx = \int_{\mathbf{R}^n} T(f_j)(x)\varphi_j(x)\,dx. \tag{4.1.19}$$

By the boundedness of T, it follows that $T(f_j)$ converges to $T(f)$ in L^2 and thus as $j \to \infty$ we have

$$\int_{\mathbf{R}^n} T(f_j)(x)\varphi_j(x)\,dx \to \int_{\mathbf{R}^n} T(f)(x)\varphi(x)\,dx. \tag{4.1.20}$$

Now write $f_j(y)\varphi_j(x) - f(y)\varphi(x) = (f_j(y) - f(y))\varphi_j(x) + f(y)(\varphi_j(x) - \varphi(x))$ and observe that

$$\left| \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} K(x,y)f(y)(\varphi_j(x)-\varphi(x))\,dy\,dx \right| \le Ac^{-n}\|f\|_{L^1}\|\varphi_j - \varphi\|_{L^1} \to 0\,,$$

since $\|\varphi_j - \varphi\|_{L^1} \le C\|\varphi_j - \varphi\|_{L^2} \to 0$ as $j \to \infty$, and

$$\left| \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} K(x,y)(f_j(y)-f(y))\varphi_j(x)\,dy\,dx \right| \le Ac^{-n}\|f_j - f\|_{L^1}\|\varphi\|_{L^1} \to 0\,,$$

as $j \to \infty$. Combining these facts with (4.1.19) and (4.1.20) we obtain

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} K(x,y)f_j(y)\varphi_j(x)\,dy\,dx \to \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} K(x,y)f(y)\varphi(x)\,dy\,dx$$

as $j \to \infty$ and proves the validity of (4.1.17). Note that the double integral on the right is absolutely convergent and bounded by $A\,(2c)^{-n}\|f\|_{L^1}\|\varphi\|_{L^1}$.

To prove (4.1.18) we fix a compactly supported and bounded function f and we pick f_j as before. Then $T(f_j)$ converges to $T(f)$ in L^2 and thus a subsequence $T(f_{j_l})$ converges pointwise on $\mathbf{R}^n \setminus E(f)$, for some measurable set $E(f)$ with $|E(f)| = 0$. Given $x_0 \notin E(f) \cup \operatorname{supp} f$ we have

$$T(f_{j_l})(x_0) = \int_{\mathbf{R}^n} K(x_0,y)f_{j_l}(y)\,dy$$

and letting $l \to \infty$ we obtain (4.1.18) since $T(f_{j_l})(x_0) \to T(f)(x_0)$ and

$$\left| \int_{\mathbf{R}^n} K(x_0,y)f_{j_l}(y)\,dy - \int_{\mathbf{R}^n} K(x_0,y)f(y)\,dy \right| \le Ac^{-n}\|f_{j_l} - f\|_{L^1} \to 0\,.$$

as $l \to \infty$. Thus (4.1.18) holds. $\square$

We now define truncated kernels and operators.

Definition 4.1.10. Given a kernel K in $SK(\delta,A)$ and $\varepsilon > 0$, we define the *truncated kernel*

$$K^{(\varepsilon)}(x,y) = K(x,y)\chi_{|x-y|>\varepsilon}\,.$$

Given a continuous linear operator T from $\mathscr{S}(\mathbf{R}^n)$ to $\mathscr{S}'(\mathbf{R}^n)$ and $\varepsilon > 0$, we define the *truncated operator* $T^{(\varepsilon)}$ by

$$T^{(\varepsilon)}(f)(x) = \int_{\mathbf{R}^n} K^{(\varepsilon)}(x,y)\,f(y)\,dy$$