An alternative characterization of *BMO* can be obtained in terms of commutators of singular integrals. Precisely, we have that the commutator $[b,T](f)$ is L^p bounded for $1 < p < \infty$ if and only if the function b is in *BMO*. The sufficiency of this result (Theorem 3.5.6) is due to Coifman, Rochberg, and Weiss [96], who used it to extend the classical theory of H^p spaces to higher dimensions. The necessity was obtained by Janson [203], who also obtained a simpler proof of the sufficiency. The exposition in Section 3.5 is based on the article of Pérez [295]. This approach is not the shortest available, but the information derived in Lemma 3.5.5 is often useful; for instance, it is used in the substitute of the weak type $(1,1)$ estimate of Exercise 3.5.4. The inequality (3.5.3) in Lemma 3.5.4 can be reversed as shown by Pérez and Wheeden [297]. Weighted L^p estimates for the commutator in terms of the double iteration of the Hardy–Littlewood maximal operator can be deduced as a consequence of Lemma 3.5.5; see the article of Pérez [296].

Orlicz spaces were introduced by Birbaum and Orlicz [39] and further elaborated by Orlicz [287], [288]. For a modern treatment one may consult the book of Rao and Ren [302]. Bounded mean oscillation with Orlicz norms was considered by Strömberg [332].

The space of functions of vanishing mean oscillation (*VMO*) was introduced by Sarason [309] as the set of integrable functions f on $\mathbf{T}^1$ satisfying $\lim_{\delta\to 0}\sup_{I:\,|I|\le\delta}|I|^{-1}\int_I|f-\mathrm{Avg}_I f|\,dx=0$. This space is the closure in the *BMO* norm of the subspace of $BMO(\mathbf{T}^1)$ consisting of all uniformly continuous functions on $\mathbf{T}^1$. One may define $VMO(\mathbf{R}^n)$ as the space of functions on $\mathbf{R}^n$ that satisfy $\lim_{\delta\to 0}\sup_{Q:\,|Q|\le\delta}|Q|^{-1}\int_Q|f-\mathrm{Avg}_Q f|\,dx=0$, $\lim_{N\to\infty}\sup_{Q:\,\ell(Q)\ge N}|Q|^{-1}\int_Q|f-\mathrm{Avg}_Q f|\,dx=0$, and $\lim_{R\to\infty}\sup_{Q:\,Q\cap B(0,R)=\emptyset}|Q|^{-1}\int_Q|f-\mathrm{Avg}_Q f|\,dx=0$; here I denotes intervals in $\mathbf{T}^1$ and Q cubes in $\mathbf{R}^n$. Then $VMO(\mathbf{R}^n)$ is the closure of the the space of continuous functions that vanish at infinity in the $BMO(\mathbf{R}^n)$ norm. One of the important features of $VMO(\mathbf{R}^n)$ is that it is the predual of $H^1(\mathbf{R}^n)$, as was shown by Coifman and Weiss [97]. As a companion to Corollary 3.4.10, certain singular integral operators (such as the Riesz transforms) can be shown to map the space of continuous functions that vanish at infinity into *VMO*. We refer to the article of Dafni [109] for a short and elegant exposition of these results as well as for a local version of the *VMO*-H^1 duality.